## Section A: Pure Mathematics [40 marks]

**1** **(a)** The functions f and g are defined as follows

$$\mathrm{f}: x \mapsto \sqrt{7-x}, \quad \text{where } x \in \mathbb{R},\ x < 7,$$
$$\mathrm{g}: x \mapsto (x-2)^4 + 2x^2, \quad \text{where } x \in \mathbb{R},\ x > -2.$$

**(i)** Show that the composite function gf exists and define gf in a similar form. [2]

**(ii)** Find the range of gf. [2]

**(b)** The functions k and kh are defined by

$$\mathrm{k}: x \mapsto x-5, \quad \text{where } x \in \mathbb{R},$$
$$\mathrm{kh}: x \mapsto x^2 + a, \quad \text{where } x \in \mathbb{R},\ x > \sqrt{5},$$

where $a$ is a positive constant.

Leave your answers for the following parts, **(b)(i)** and **(ii),** in terms of $a$.

**(i)** Find h($x$). [2]

**(ii)** Explain why $\mathrm{h}^{-1}$ exists. Hence or otherwise, find $\mathrm{h}^{-1}(x)$ and state the domain of $\mathrm{h}^{-1}$. [3]

**2** The line $l_1$ has equation $\mathbf{r} = (10 - 3\lambda)\mathbf{i} + (4\lambda - 3)\mathbf{j} + (1 - \lambda)\mathbf{k}$ where $\lambda$ is a real parameter and the line $l_2$ has equation $\dfrac{3-x}{6} = \dfrac{y+5}{7} = \dfrac{2-z}{2}$.

**(i)** Determine whether lines $l_1$ and $l_2$ intersect. [2]

**(ii)** The position vector of the points $A$ and $B$ on line $l_1$ are such that $\lambda = -1$ and $\lambda = 1$ respectively. Find the length of projection of $\overrightarrow{AB}$ onto $l_2$. [2]

**(iii)** Find an equation of the plane $p$ that includes the origin $O$ and the line $l_1$. [2]

**(iv)** Find the cartesian equations of the planes such that the perpendicular distance from each plane to $p$ is $\sqrt{1011}$. [2]

**(v)** Find the exact coordinates of the point of intersection between the line $l_2$ and the plane $p$. [3]

**3** A stone is released from a drone for an experiment. The stone falls vertically and the distance, $x$ metres, that the stone has fallen in time $t$ seconds is measured. It is given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

**(i)** The motion of the stone is modelled by the differential equation

$$10\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 = 100.$$

By substituting $v = \frac{dx}{dt}$, show that the differential equation can be written as $\frac{dv}{dt} = \frac{100 - v^2}{10}$. [1]

**(ii)** Find $v$ in terms of $t$ and hence find $x$ in terms of $t$. [4]

[You may use the result : $\int \frac{e^{2x} - 1}{e^{2x} + 1}\, dx = \ln\left(e^x + e^{-x}\right) + c$, where $c$ is an arbitrary constant.]

**(iii)** A second model for the motion of the stone is suggested, given by the differential equation

$$\frac{d^2x}{dt^2} = 10e^{-t}.$$

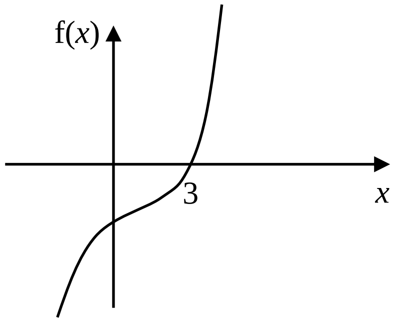
Find $x$ in terms of $t$ for this model. [3]

**(iv)** Given that the stone is dropped from a height of 100 m, determine which model will predict that the stone will reach the ground first. [1]

**4** **(a)** **(i)** Using $e^{i\theta} = \cos\theta + i\sin\theta$ and the trigonometry formulas, show that $\left(e^{i\theta_1}\right)\left(e^{i\theta_2}\right) = e^{i(\theta_1+\theta_2)}$. [2]

**(ii)** Hence find the least positive integer value of $n$ such that the complex number $\left(ie^{i\frac{\pi}{5}}\right)^n$ is
purely imaginary. [3]

**(b)** The graph of $f(x) = x^3 - 7x^2 + 17x - 15$ is shown below. It cuts the $x$-axis at $x = 3$.




**(i)** Without the use of a graphing calculator, show that the roots of the equation of $f(x) = 0$ are $m$, $z_0$ and $z_1$ where $m$ is a real constant and $z_0$ and $z_1$ are complex constants to be determined. [4]

**(ii)** The complex number $w$ is such that $w = z_0 + \lambda(z_0 - z_1)$ where $\lambda$ is a real number. By considering an Argand diagram or otherwise, find the least value of $|w|$ when $\lambda$ varies. [2]

## Section B: Probability and Statistics [60 marks]

**5** There are four houses in a particular school. A sports carnival is organised in the school.

Each house sends 2 students to compete in a 100-metre race during the sports carnival.

**(i)** Find the number of ways to arrange the students in a row at the starting line, if students from the same house must stand next to each other. [2]

**(ii)** A house is considered the winner if its 2 students are in the top 3 placing. Assuming the students cross the finish line one after another, find the number of ways for a particular house to be the winner and the last 2 students belong to the same house. [3]

Each house has a captain and 4 teachers.

**(iii)** The captain and 3 teachers from each house are selected to play a game in a circle. Find the number of ways to arrange them if the teachers from the same house must stand next to each other, and the captains must not stand next to one another. [3]

**6** Arthur rolls a biased die and a fair die, each with 6 sides labelled 1 to 6. The probability of scoring $x$ on the biased die is $kx$, where $k$ is a constant.

**(i)** Find the value of $k$. [1]

The random variables $B$ and $F$ are the scores from a single roll on the biased die and fair die respectively. The table below shows the probabilities of obtaining the respective values of $B$ and $F$.

| **B** \\ **F** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| **1** | $\frac{1}{6}k$ | $\frac{1}{6}k$ | $\frac{1}{6}k$ | | $\frac{1}{6}k$ | $\frac{1}{6}k$ |
| **2** | $\frac{1}{3}k$ | $\frac{1}{3}k$ | $\frac{1}{3}k$ | $\frac{1}{3}k$ | | $\frac{1}{3}k$ |
| **3** | $\frac{1}{2}k$ | | $\frac{1}{2}k$ | $\frac{1}{2}k$ | $\frac{1}{2}k$ | $\frac{1}{2}k$ |
| **4** | $\frac{2}{3}k$ | $\frac{2}{3}k$ | $\frac{2}{3}k$ | $\frac{2}{3}k$ | $\frac{2}{3}k$ | |
| **5** | $\frac{5}{6}k$ | $\frac{5}{6}k$ | | $\frac{5}{6}k$ | $\frac{5}{6}k$ | $\frac{5}{6}k$ |
| **6** | | $k$ | $k$ | $k$ | $k$ | $k$ |

**(ii)** Complete the table by filling in the missing probabilities. [1]

**(iii)** Show that $\mathrm{P}(B > F) = \frac{5}{9}$. [2]

$S$ is the random variable denoting the product of the two scores.

**(iv)** Find $\mathrm{P}(S = 5 \mid B > F)$. [2]

**(v)** Arthur observes that $\mathrm{P}(S = 5) = k$, corresponding to the cases $\{B = 5,\ F = 1\}$ and $\{B = 1,\ F = 5\}$. Write down two other values of $s$ such that $\mathrm{P}(S = s) = k$, and the corresponding cases for each value of $s$. [2]

**7** The manager of a factory claims that his workers take an average of 3.5 minutes to complete a particular task. Sixty workers are randomly chosen and the time taken, $x$ minutes, of each worker is measured. The results are summarised by $\sum x = 216, \quad \sum (x-\bar{x})^2 = 6.4$.

**(i)** Calculate unbiased estimates of the population mean and variance of the time taken. [2]

**(ii)** Test, at the 1% level of significance, whether the manager has understated the average time taken. You should state your hypotheses and define any symbols you use. [5]

**(iii)** Explain the meaning of 'at the 1% level of significance' in the context of the question. [1]

Another $n$ workers in the same factory are randomly chosen and combined with the sixty workers into a single sample. For the combined single sample, it is known that the mean time taken is 3.55 minutes.

**(iv)** Given that the manager's claim is not rejected for a 2-tail test at the 10% level of significance, find the largest value of $n$. You may assume that the unbiased estimate of the population variance of the time taken remains unchanged. [4]

**8** An air force consists of a large number of fighter planes. Each plane carries and releases 8 bombs at one time to destroy a designated enemy target. Based on past records, the proportion of bombs hitting an enemy target is $p$.

**(i)** State, in the context of the question, two assumptions needed for the number of bombs from a plane to hit an enemy target to be well modelled by a binomial distribution. [2]

Assume now that, $X$, the number of bombs from a plane hitting an enemy target has a binomial distribution.

**(ii)** If $\mathrm{E}(X) + \mathrm{Var}(X) = 6.5$, show that $p = 0.567$ correct to 3 decimal places. [2]

**(iii)** Out of 30 randomly chosen planes that were despatched, estimate the probability that the average number of bombs per plane hitting their targets is less than 5. [2]

A mission is considered a success if an enemy bridge is destroyed.

**(iv)** 4 bombs must hit a bridge before it is destroyed. Find the probability of a successful mission for a plane. [2]

**(v)** 20 planes are randomly despatched to destroy 1 bridge each. Find the probability that at least 13 but fewer than 19 planes' missions are successful. [2]

**9** **In this question you should state the parameters of any distribution you use**.

A company operates a call centre which offers call-in technical support service for their customers. The customer calling in will be placed on a queue before a technical support officer can render the service.

The waiting time of a randomly chosen call, in minutes, is normally distributed with mean and standard deviation as given in the following table.

| | Mean | Standard deviation |
|---|---|---|
| Peak hours | 9 | 1.5 |
| Off-peak hours | 3 | 0.5 |

The time taken by the technical officers to render the service to a randomly chosen customer, in minutes, is normally distributed with mean and standard deviation as given in the following table.

| | Mean | Standard deviation |
|---|---|---|
| Minor problem | 15 | 5 |
| Intermediate problem | 30 | 10 |

The time taken to resolve the problem for a customer is the sum of the waiting time and the time taken to render the service. You may assume that the waiting and service times are independent of each other.

**(i)** Find the probability that a randomly chosen customer who calls in during off-peak hours for a minor problem can resolve his problem in less than 20 minutes. [2]

**(ii)** Find the probability that out of 3 randomly chosen customers who call in during off-peak hours for a minor problem, at least one customer can resolve his problem in less than 20 minutes. [2]

**(iii)** A group of 5 randomly chosen customers call during off-peak hours and face a minor problem. Another group of 5 randomly chosen customers call during peak hours and face an intermediate problem. Find the probability that the difference between the average time taken by each group of customers to resolve the problem is at most 30 minutes. [4]

The company decides to send its technical support officers for training. After the training, the time taken by the technical support officers to render the service for an intermediate problem has the distribution $N(\mu, \sigma^2)$. It is found that the time spent on 8% of the problems serviced by the officers is less than 16 minutes, while 20% of the problems require more than 34 minutes.

**(iv)** Find the value of $\mu$ and of $\sigma$, giving your answers correct to 3 significant figures. Justify whether the training was effective. [4]

**10** In every round of a television quiz, a contestant is given one sample each from four different brands of instant noodles. The task is to match the sample to its brand correctly in a blind taste test. Since the samples look and taste similar, it may be assumed that the contestant guesses randomly.

**(i)** Write down the total number of distinct ways that a contestant can make a guess. [1]

**(ii)** Show that the number of distinct ways for a contestant to correctly guess exactly two out of the four brands is 6. [1]

The random variable $X$ represents the number of samples which a contestant matches correctly to its respective brand.

The table shows the probability distribution of $X$.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\mathrm{P}(X = x)$ | $p$ | $\frac{1}{3}$ | $q$ | 0 | $r$ |

**(iii)** Explain why $\mathrm{P}(X = 3) = 0$. [1]

**(iv)** Find the values of $q$ and $r$. Hence show that $p = \frac{3}{8}$. [2]

For each sample matched correctly to its brand in each round, a contestant is given \$50.

**(v)** Find the expectation and variance of the prize money won by a contestant in the first round of the contest. [3]

A contestant who matches correctly all four samples to their brands in the first round will advance to play the second round, otherwise, the quiz will terminate at the end of the first round. Thus a contestant can play at most 2 rounds in the quiz.

**(vi)** Find the expected total prize money a contestant will earn. [2]