CJC 2019 H2 Mathematics

Prelim P2

Section A: Pure Mathematics (40 Marks)

1 The function h is given by $\mathrm{h}(x) = ax^3 + bx^2 + \dfrac{x}{c}$, $x \in \mathbb{R}$, where a, b and c are real constants.

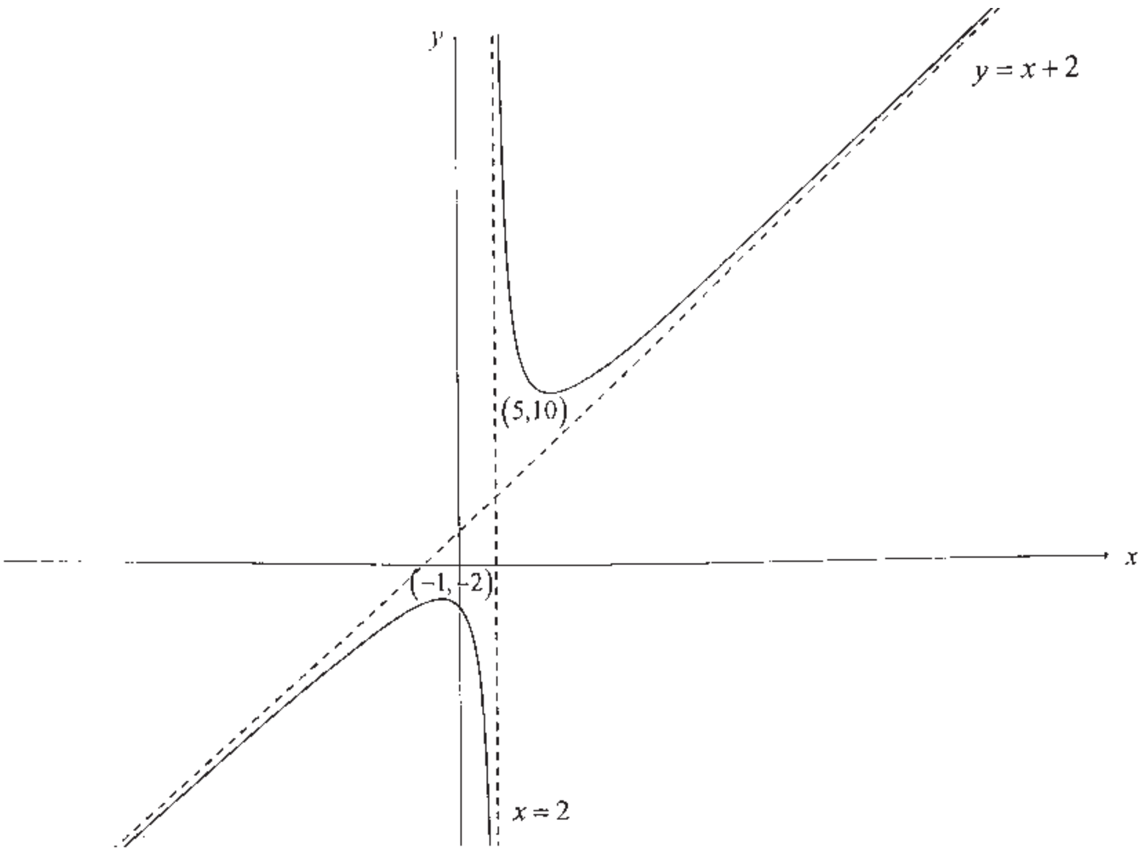
The graph of $y = \mathrm{h}(x)$ passes through the point $\left(1, \dfrac{13}{4}\right)$. The point $(-8, 642)$ lies on the graph of $y = \mathrm{h}(|x|)$ and the point $\left(4, \dfrac{1}{97}\right)$ lies on the graph of $y = \dfrac{1}{\mathrm{h}(x)}$. Find the values of a, b and c. [4]

2 Given that $k\mathbf{p} = (\mathbf{p}.\mathbf{q})\mathbf{q}$ where k is a positive constant, $\mathbf{p}$ and $\mathbf{q}$ are non – zero vectors.

(i) What is the geometrical relationship between $\mathbf{p}$ and $\mathbf{q}$? [1]

(ii) Find $|\mathbf{q}|$ in terms of k. [3]

3 The diagram below shows the graph of $y = \mathrm{f}(x)$. The curve has a minimum point $(5,10)$ and a maximum point $(-1,-2)$. The lines $x = 2$ and $y = x + 2$ are asymptotes of the graph.



(i) Sketch the curve $y = \mathrm{f}'(x)$, indicating clearly the coordinates of the points where the graph crosses the x axis and the equations of any asymptotes. [3]

(ii) State the range of values of x for which the graph of $y = \mathrm{f}(x)$ is

(a) strictly decreasing, [1]

(b) concave upwards. [1]

4 The function f is defined by $\mathrm{f}: x \to x|x-3|, \quad x \in \mathbb{R}, 2 \le x < 3$.

(i) Explain, with the aid of a sketch, why the inverse function f^{-1} exists.

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

It is given that $\mathrm{g}(x) = \begin{cases} 2x-2, & 0 \le x \le 2 \\ \dfrac{10-2x}{x+1}, & 2 < x < 4. \end{cases}$

(iii) Sketch the graph of $y = \mathrm{g}(x)$. [2]

(iv) Find gf (x). [2]

(v) Find the range of gf. [1]

5 A curve C has parametric equations $x = 1 - \sin\theta, \quad y = \theta + \cos\theta$, where $-\pi \le \theta \le \dfrac{\pi}{2}$.

(i) Sketch the graph of C, stating the exact coordinates of the end points. [2]

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ. What can be said about the tangent to C as $\theta \to -\dfrac{\pi}{2}$?

[3]

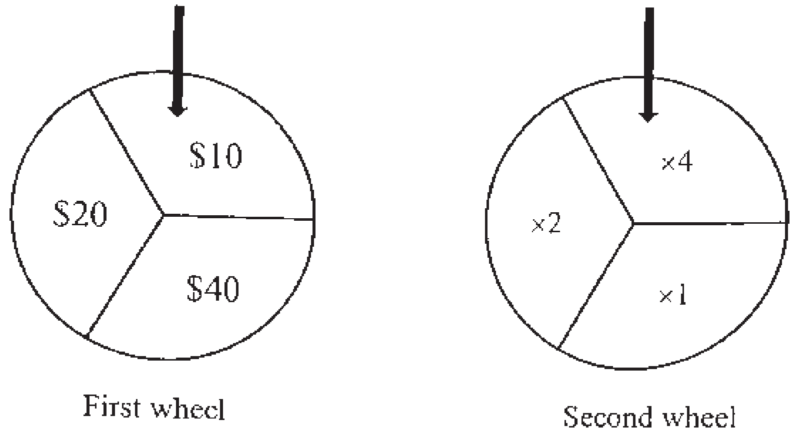
(iii) A point P on C has parameter p, where $0 < p < \dfrac{\pi}{2}$. Show that the normal to C at P crosses the y axis at point Q with coordinates $(0, p)$. [3]

(iv) Show that the area of region bounded by C, the normal to C at point P and the y axis is given by $a\pi + bp + c\cos p$, where a, b and c are to be determined. [6]

(v) The normal to C at P also crosses the x axis at point R. Find a Cartesian equation of the locus of the midpoint of QR as p varies. [3]

Section B: Probability and Statistics (60 marks)

6 To raise its profile, ABC Supermart devised a publicity where, if a customer has purchases of $\$R$ or more in a single receipt, he is qualified to participate in a sure-win "Lucky Spin" game. In the game, the customer spins the first wheel (which is divided into 3 equal parts) to see how much money he wins, then he spins a second wheel to see how much his winning is multiplied by.



For example, based on the illustration above, if the result from the first spin is "$10", and the result of the second spin is " × 4", then the person playing the game would have received $40.

Let X denote the amount won from playing the game.

(i) Tabulate the probability distribution of X. [2]

(ii) ABC Supermart makes a profit of 40% on all purchases made by customers. It wishes to use the profits generated from every qualifying receipt to offset the amounts given away in the game. Determine the value of $\$R$ that the Supermart needs to set for a customer to qualify to play the game. Give your answer correct to the nearest dollar. [3]

(iii) Find the value of σ, the standard deviation of X. Hence find $P(X < \sigma)$. [3]

7 An experiment is being carried out to study the correlation between the solubility of sugar in water and the temperature of the water. The amount of sucrose x (in grams) dissolved in 100 ml of water for different water temperatures T (in degree Celsius) is recorded. The results shown in the table. Unfortunately, one of the values of x was accidentally deleted from the records later on and it is indicated by k as shown below.

T	0	20	40	60	80	100
X	179	204	241	k	363	487

It is given that the equation of the regression line of x on T is $x = 2.94857T + 146.238$. Show that $k = 288$.

(i) Draw a scatter diagram to illustrate the data, labelling the axes clearly. [1]

The value of the product moment correlation coefficient between x and T is 0.959. It is thought that a model given by the formulae $\ln x = a + bT$ may also be a suitable fit to the data where a and b are constants.

(ii) Calculate the least square estimates of a and b and find the value of the product moment correlation coefficient between T and $\ln x$. [2]

(iii) Use your answers to parts (i) and (ii) to explain which of $x = 2.94857T + 146.238$ or

$\ln x = a + bT$ is the better model. [2]

(iv) Hence, predict the value of T for which $x = 300$. Comment on the reliability of your prediction. [2]

8 Cynthia is a skilled pistol shooter who hits the bullseye of the target 70% of the time. During her training sessions, she shoots a series of 15 shots on a target before changing to a new target.

A series is considered "good" if she is able to hit the bullseye at least 11 times on the target.

(i) Find the probability that Cynthia is able to obtain a good series. [2]

Cynthia is able to end the training session early if she is able to obtain at least 6 good series in n attempts.

(ii) Find the least value of n such that she can be at least 99% certain of being able to end the training session early. [4]

Suppose Cynthia has completed 40 of the fifteen-shot series,

(iii) estimate the probability that she has, on average, hit the bullseye more than 10 times per series. [3]

9 The security guard of a particular school claims that the average speed of the cars in the school compound is greater than the speed limit of 25 km/h. To investigate the security guard's claim, the traffic police randomly selected 50 cars and the speed was recorded. The total speed and the standard deviation of the 50 cars are found to be 1325 km/h and 7.75 km/h respectively.

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Test at 5% level of significance whether there is sufficient evidence to support the security guard's claim. [4]

It is now known that the speed of the cars is normally distributed with mean 25 km/h and standard deviation of 6 km/h.

(iii) A new sample of n cars is obtained and the sample mean is found to be unchanged. Using this sample, the traffic police conducts another test at 10% level of significance and concludes that the security guard's claim is valid. Find the set of values n can take. [3]

(iv) What is the speed exceeded by 75% of the cars? [1]

10 The Mathematics examination score of a randomly chosen student in Group A is X, where X follows a normal distribution with mean 55 and standard deviation σ. The Mathematics examination score of a randomly chosen student in Group B is Y, where Y follows a normal distribution with mean 45 and standard deviation 10.

(i) It is known that $2P(X > 45) = 5P(X > 65)$. Show that $\sigma = 17.7$, correct to 3 significance figures. [3]

(ii) Find the probability that total score of 3 randomly selected students from Group A differ from 4 times the score of a randomly selected student from Group B by at most 10. [4]

(iii) Find the probability that the mean score of 3 randomly selected students from Group A and 4 randomly selected students from Group B is at least 50. [3]

The Mathematics examination scores of students in Group C are found to have a mean of 40 and standard deviation of 25. Explain why the examination scores of students in Group C are unlikely to be normally distributed. [1]

11 A palindrome is a string of letters or digits that is the same when you read it forwards or backwards. For example: HHCCHH, RACECAR, STATS are palindromes.

A computer is instructed to use any of the letters **R, O, F, L** to randomly generate a string of 5 letters. Repetition of any letter is allowed, but the string cannot contain only one letter. For example, RRRRR is not allowed.

Events A and B are defined as follows:

A: the string generated contains 2 distinct letters

B: the string generated is a palindrome

(i) Find $P(A)$ and $P(B)$. [5]

(ii) Find $P(A \cap B)$ and hence determine if A and B are independent. [3]

(iii) Find the probability that the string generated either contains 2 distinct letters, or that it is a palindrome, or both. [2]

(iv) Find the probability that the string generated contains 2 distinct letters, given that it is a palindrome. [2]