Jurong Pioneer Junior College
H2 Mathematics Preliminary Exam P1 Solution

Q1

(i)

Let $u_m = am^3 + bm^2 + cm + d$

$u_1 = a + b + c + d = -13$

$u_2 = 8a + 4b + 2c + d = -12.8$

$u_3 = 27a + 9b + 3c + d = 1.8$

$u_4 = 64a + 16b + 4c + d = 38$

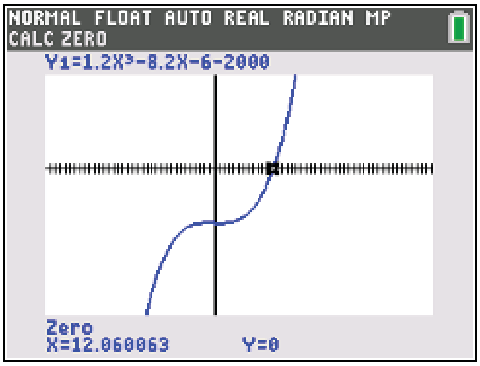
Using GC, $a = 1.2$, $b = 0$, $c = -8.2$, $d = -6$

$u_m = 1.2m^3 - 8.2m - 6$

(ii)

$u_m > 2000$

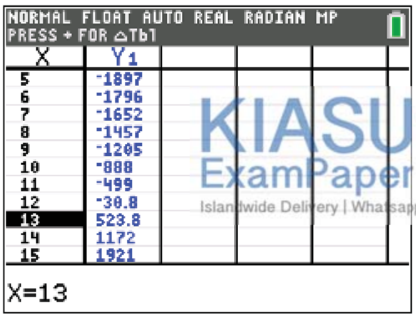
Let $y = 1.2m^3 - 8.2m - 6 - 2000$



From GC graphing, $m > 12.06$

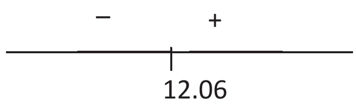
$m \geq 13$ where m is an integer

Or : From GC table,



$m \geq 13$ where m is an integer

Or : GC poly root finder



$m > 12.06$

$m \geq 13$ where m is an integer

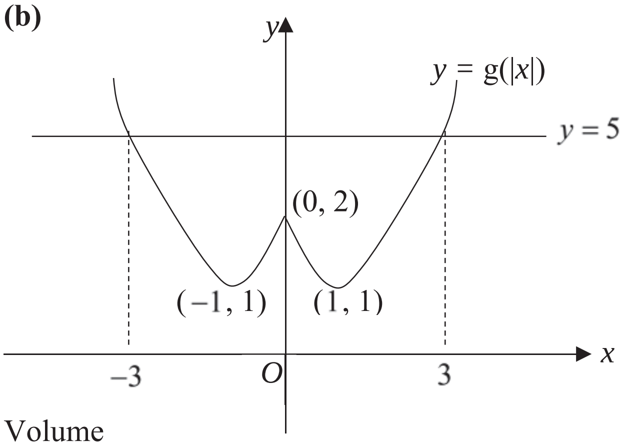
Q2

(a)

$$y = \frac{3x-1}{x-2} = 3 + \frac{5}{x-2}$$

- Translation of 2 units in the positive x direction.
- Scaling parallel to the y-axis by a scale factor of 5.
- Translation of 3 units in the positive y direction.

(b)



Volume

$$= 2\left[\pi(5)^2(3) - \pi\int_0^3 \left(x^2 - 2x + 2\right)^2 \, dx\right]$$

$$= 373 \text{(3sf)}$$

or

Volume

$$= \pi(5)^2(6) - \pi\int_{-3}^3 \left(|x|^2 - 2|x| + 2\right)^2 \, dx$$

$$= 373 \text{(3sf)}$$

Note :

Cannot use

$$\pi(5)^2(6) - \pi\int_{-3}^3 \left(x^2 - 2x + 2\right)^2 \, dx$$

Cannot use

$\pi(5)^2(6) - \pi\int_{-3}^3 \left(|x|^2 - 2|x| + 2\right)^2 \, dx$ if question ask for exact.

Q3

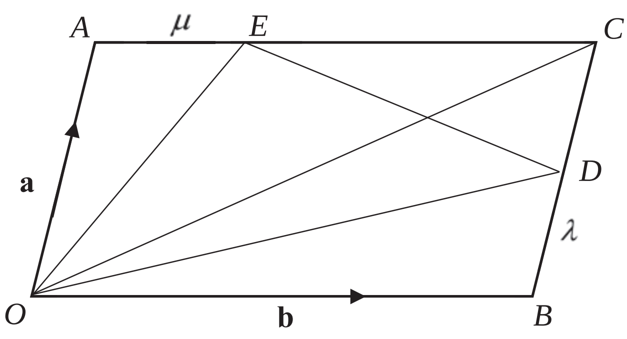
(i)

$\overrightarrow{OD} = \mathbf{b} + \lambda\mathbf{a}$

$\overrightarrow{OE} = \mathbf{a} + \mu\mathbf{b}$

area of triangle OCE

$$= \frac{1}{2}\left|\overrightarrow{OC} \times \overrightarrow{EC}\right|$$



$= \frac{1}{2}\left|(\mathbf{a}+\mathbf{b})\times(1-\mu)\mathbf{b}\right|$

$= \frac{1-\mu}{2}|\mathbf{a}\times\mathbf{b}|$ since $0<\mu<1$

area of triangle *ODE*

$= \frac{1}{2}\left|\overrightarrow{OD}\times\overrightarrow{OE}\right|$

$= \frac{1}{2}\left|(\mathbf{b}+\lambda\mathbf{a})\times(\mathbf{a}+\mu\mathbf{b})\right|$

$= \frac{1}{2}\left|(\mathbf{b}\times\mathbf{a})+\lambda\mu(\mathbf{a}\times\mathbf{b})\right|$

$= \frac{1}{2}\left|(\mathbf{b}\times\mathbf{a})-\lambda\mu(\mathbf{b}\times\mathbf{a})\right|$

$= \frac{1-\lambda\mu}{2}|\mathbf{b}\times\mathbf{a}|$

$= \frac{1-\lambda\mu}{2}|\mathbf{a}\times\mathbf{b}|$ since $0<\mu<1$ and $0<\lambda<1$

area of triangle *ODE* = *k* (area of triangle *OCE*)

$\frac{1-\lambda\mu}{2}|\mathbf{a}\times\mathbf{b}| = k\left(\frac{1-\mu}{2}\right)|\mathbf{a}\times\mathbf{b}|$

$1-\lambda\mu = k(1-\mu)$

$k = \frac{1-\lambda\mu}{1-\mu}$

Note :

- explanation $0<\mu<1$ and $0<\lambda<1$ should be seen.

- For area, must have modulus if written as $\lambda\mu-1$ and $\mu-1$. i.e $|\lambda\mu-1|$, $|\mu-1|$.

- For *k*, if written as $k = \left|\frac{1-\lambda\mu}{1-\mu}\right|$, must proceed to

$k = \frac{1-\lambda\mu}{1-\mu}$

or

$k = -\left(\frac{1-\lambda\mu}{1-\mu}\right)$ (rejected as $k>0$)

(ii)

Given $OF:FC = 6:1$ and $DF:FE = 3:4$.

By Ratio Theorem,

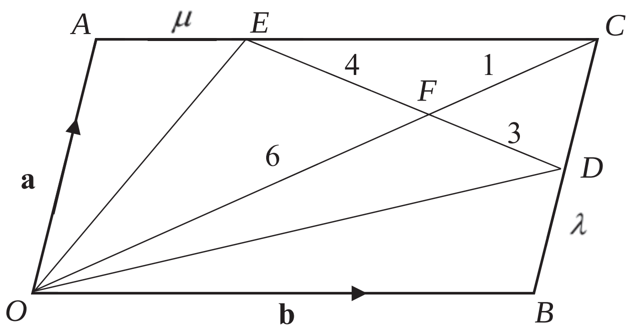
$\overrightarrow{OF} = \frac{3\overrightarrow{OE}+4\overrightarrow{OD}}{7}$

$\frac{6}{7}(\mathbf{a}+\mathbf{b}) = \frac{1}{7}[3(\mathbf{a}+\mu\mathbf{b})]+\frac{1}{7}[4(\mathbf{b}+\lambda\mathbf{a})]$

$\mathbf{a}$: $6 = 3+4\lambda \Rightarrow \lambda = \frac{3}{4}$

$\mathbf{b}$: $6 = 3\mu+4 \Rightarrow \mu = \frac{2}{3}$

$k = \frac{1-\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$



Q4

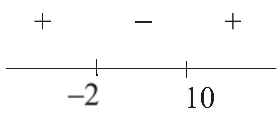
(i)

Consider $y=\frac{x^2+5}{x-2}$ and the line $y=k$.

$$k=\frac{x^2+5}{x-2}$$
$$kx-2k=x^2+5$$
$$x^2-kx+2k+5=0$$

For $y=\frac{x^2+5}{x-2}$ and the line $y=k$ not to intersect,

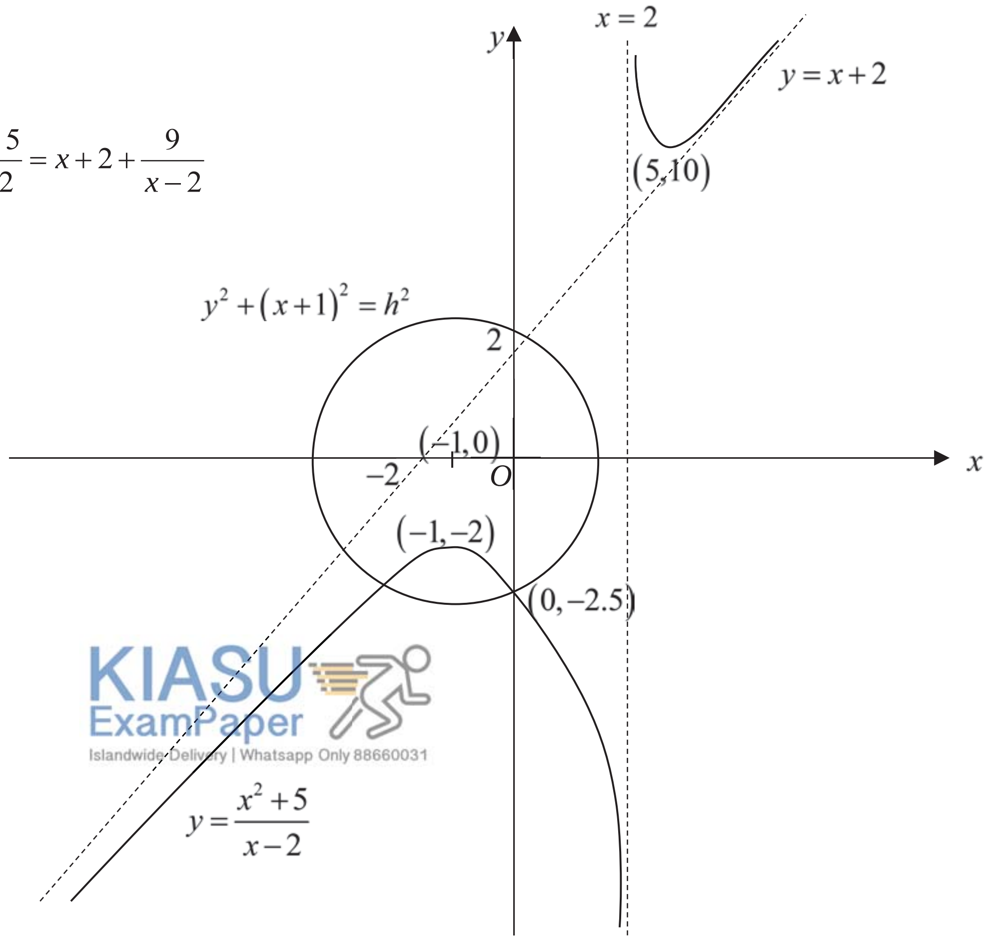
$$k^2-4(2k+5)<0$$
$$k^2-8k-20<0$$
$$(k-10)(k+2)<0$$
$$-2<k<10$$
$$-2<y<10$$



C cannot lie between -2 and 10.

(ii)

$$y=\frac{x^2+5}{x-2}=x+2+\frac{9}{x-2}$$



(iii)

$$\left(x^2+5\right)^2+\left(x+1\right)^2\left(x-2\right)^2=h^2\left(x-2\right)^2$$

$$\frac{\left(x^2+5\right)^2}{\left(x-2\right)^2}+\left(x+1\right)^2=h^2$$

$$y^2+\left(x+1\right)^2=h^2$$

Consider distance from centre of circle to *y*-intercept of *C*

$$|h|=\sqrt{\left(-1-0\right)^2+\left(0-\left(-2.5\right)\right)^2}=\frac{1}{2}\sqrt{29}\ \text{ or } 2.69$$

$$h^2>\frac{29}{4}$$

$$h<-\frac{1}{2}\sqrt{29}\ \text{ or }\ h>\frac{1}{2}\sqrt{29}$$

Q5

(a)(i)

$$w_1^{\,3}=\left(-3+\sqrt{5}\,\mathrm{i}\right)^3$$

$$=\left(-3\right)^3+3\left(-3\right)^2\left(\left(\sqrt{5}\right)\mathrm{i}\right)+3\left(-3\right)\left(\left(\sqrt{5}\right)\mathrm{i}\right)^2+\left(\left(\sqrt{5}\right)\mathrm{i}\right)^3$$

$$=-27+27\sqrt{5}\,\mathrm{i}+45-5\sqrt{5}\,\mathrm{i}$$

$$=18+22\sqrt{5}\,\mathrm{i}$$

(ii)

Since $w_1=-3+\sqrt{5}\,\mathrm{i}$ is a root,

$$4w_1^{\,3}+pw_1^{\,2}+qw_1-14=0$$

$$4(18+22\sqrt{5}\,\mathrm{i})+p\left(-3+\sqrt{5}\,\mathrm{i}\right)^2+q\left(-3+\sqrt{5}\,\mathrm{i}\right)-14=0$$

$$4(18+22\sqrt{5}\,\mathrm{i})+p\left(9-6\sqrt{5}\,\mathrm{i}-5\right)+q\left(-3+\sqrt{5}\,\mathrm{i}\right)-14=0$$

$$72+4p-3q-14+\left(88\sqrt{5}-6\sqrt{5}p+\sqrt{5}q\right)\mathrm{i}=0$$

Comparing real and imaginary parts,

$4p-3q+58=0$ ---------(1)

$88\sqrt{5}-6\sqrt{5}p+\sqrt{5}q=0$

$\therefore 88-6p+q=0$ ------(2)

$(2)\times 3: 264-18p+3q=0$ ------(3)

Solving (1) and (3), $14p=322$

$p=23$, $q=50$.

(iii)

Since $w_1 = -3+\sqrt{5}\,\mathrm{i}$ is a root, and polynomial equation has real coeffcients, $w_1^* = -3-\sqrt{5}\,\mathrm{i}$ is also a root.

$$4w^3+23w^2+50w-14 = \left(w-\left(-3+\sqrt{5}\,\mathrm{i}\right)\right)\left(w-\left(-3-\sqrt{5}\,\mathrm{i}\right)\right)g(w)$$
$$=\left((w+3)+\sqrt{5}\,\mathrm{i}\right)\left((w+3)-\sqrt{5}\,\mathrm{i}\right)g(w)$$
$$=\left((w+3)^2-\left(\sqrt{5}\,\mathrm{i}\right)^2\right)g(w)$$
$$=\left(w^2+6w+14\right)(4w-1)$$

When $4w^3+23w^2+50w-14=0$,

Therefore, other two roots are $-3-\sqrt{5}\,\mathrm{i}$ and $\frac{1}{4}$.

(b)

$$z=-1-\sqrt{3}\ \mathrm{i} = 2\mathrm{e}^{-\frac{2}{3}\pi \mathrm{i}}$$

$$\frac{z^*}{z^n} = \frac{2\mathrm{e}^{\frac{2}{3}\pi \mathrm{i}}}{2^n \mathrm{e}^{-\frac{2}{3}n\pi \mathrm{i}}} = 2^{1-n}\mathrm{e}^{\frac{2}{3}\pi(n+1)\mathrm{i}} = 2^{1-n}\left(\cos\left(\frac{2}{3}\pi(n+1)\right)+\mathrm{i}\sin\left(\frac{2}{3}\pi(n+1)\right)\right)$$

$\frac{z^*}{z^n}$ is imaginary: $\cos\left(\frac{2}{3}\pi(n+1)\right)=0$

$$\frac{2}{3}\pi(n+1) = \frac{(2k+1)}{2}\pi, \text{ where } k\in\mathbb{Z}$$

$$n=\frac{3k}{2}-\frac{1}{4}$$

> Note :
>
> Cosine is zero when
>
> $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$......
>
> $\pm1, \pm2, \pm3$,...... odd number

Q6

(i)

$$\mathrm{f}(x) = \mathrm{e}^{2x}-9\mathrm{e}^{-2x}$$

$$\mathrm{f}'(x) = 2\mathrm{e}^{2x}+18\mathrm{e}^{-2x}$$

Since $\mathrm{e}^{2x}>0$ and $\mathrm{e}^{-2x}>0$ for all x, $\mathrm{f}'(x)=2\mathrm{e}^{2x}+18\mathrm{e}^{-2x}>0$ for all x.

(ii)

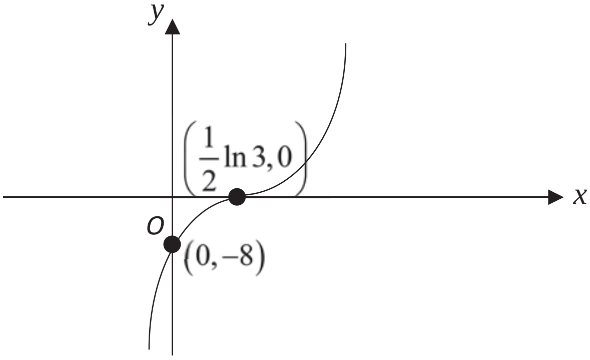
$$\mathrm{f}''(x) = 4\mathrm{e}^{2x}-36\mathrm{e}^{-2x}$$

For $y=\mathrm{f}(x)$ to be concave upward, $\mathrm{f}''(x)=4\mathrm{e}^{2x}-36\mathrm{e}^{-2x}>0 \Rightarrow \mathrm{e}^{2x}-9\mathrm{e}^{-2x}>0$ -(1)

For $f(x)>0$, $e^{2x}-9e^{-2x}>0$ which is the same as (1) (Shown)

$$e^{2x}-9e^{-2x}>0$$
$$e^{4x}-9>0$$
$$e^{4x}>9$$
$$x>\frac{1}{4}\ln 9=\frac{1}{2}\ln 3$$

(iii)



(iv)

$$\int_0^2 \left|e^{2x}-9e^{-2x}\right| dx$$
$$=-\int_0^{\frac{1}{2}\ln 3} e^{2x}-9e^{-2x}\,dx+\int_{\frac{1}{2}\ln 3}^{2} e^{2x}-9e^{-2x}\,dx$$
$$=-\left[\frac{1}{2}e^{2x}+\frac{9}{2}e^{-2x}\right]_0^{\frac{1}{2}\ln 3}+\left[\frac{1}{2}e^{2x}+\frac{9}{2}e^{-2x}\right]_{\frac{1}{2}\ln 3}^{2}$$
$$=-\left[\left(\frac{1}{2}e^{2\left(\frac{1}{2}\ln 3\right)}+\frac{9}{2}e^{-2\left(\frac{1}{2}\ln 3\right)}\right)-\left(\frac{1}{2}e^{2(0)}+\frac{9}{2}e^{-2(0)}\right)\right]+\left[\left(\frac{1}{2}e^{2(2)}+\frac{9}{2}e^{-2(2)}\right)-\left(\frac{1}{2}e^{2\left(\frac{1}{2}\ln 3\right)}+\frac{9}{2}e^{-2\left(\frac{1}{2}\ln 3\right)}\right)\right]$$
$$=-\left[\frac{3}{2}+\frac{9}{2}\left(\frac{1}{3}\right)-\frac{1}{2}-\frac{9}{2}\right]+\left[\frac{1}{2}e^{4}+\frac{9}{2}e^{-4}-\frac{3}{2}-\frac{9}{2}\left(\frac{1}{3}\right)\right]$$
$$=-1+\frac{1}{2}e^{4}+\frac{9}{2}e^{-4}$$

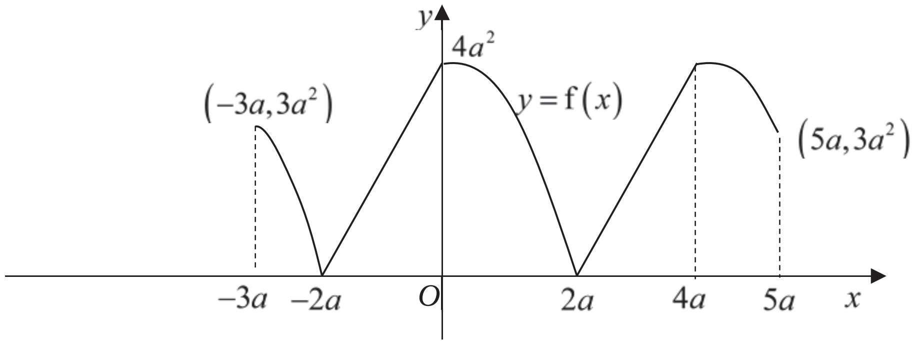
Q7

(i)

Since $f(x)=f(x+4a)$,

$$f(2019a)=f(3a)=2a(3a-2a)=2a^2$$

(ii)



(iii)

Since $R_f = [0, 4a^2] \not\subset D_g = (2a, 4a)$, gf does not exist.

(iv) Please note that g(2*a*) = 0, g(4*a*) = 2*a*, so f takes 0 to 2*a*

$fg(x) = 4a^2 - \left(\sqrt{4a^2 - (x-2a)^2}\right)^2 = (x-2a)^2$

$fg : x \mapsto (x-2a)^2, \quad 2a < x < 4a,$

(v)

$(fg)^{-1}(27) = \frac{7}{2}a$

$fg\left(\frac{7}{2}a\right) = 27$

$\left(\frac{7}{2}a - 2a\right)^2 = 27$

$\frac{9}{4}a^2 = 27$

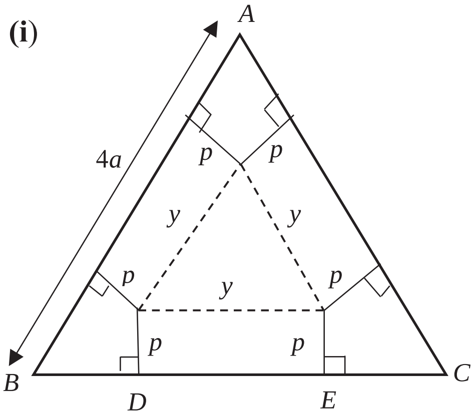
$a^2 = 12$

$a = \pm 2\sqrt{3}$

Since $a > 0$, $a = 2\sqrt{3}$

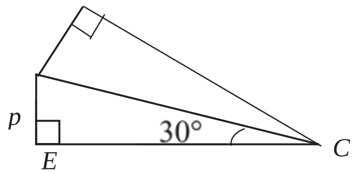
Q8

(i)



Since ABC is an equilateral triangle, $\angle ABC = \angle BCA = \angle CAB = 60^0$

length of BD = length of $CE = \dfrac{p}{\tan 30^0} = \sqrt{3}p$



Thus $y = 4a - 2\sqrt{3}p$

Volume of the prism, $V = \left(\dfrac{1}{2}y^2 \sin 60^0\right)p$

$$= \frac{1}{2}\left(4a - 2\sqrt{3}p\right)^2 \times \frac{\sqrt{3}}{2} \times p$$

$$= \frac{\sqrt{3}}{4}p\left(4a - 2\sqrt{3}p\right)^2$$

$$= \frac{\sqrt{3}}{4}p(2)^2\left(2a - \sqrt{3}p\right)^2$$

$$= \sqrt{3}p\left(2a - \sqrt{3}p\right)^2 \text{ cm}^3 \text{ (Shown)}$$

(ii)

$$\frac{dV}{dp} = \sqrt{3}\left(2a - \sqrt{3}p\right)^2 + 2\sqrt{3}p\left(2a - \sqrt{3}p\right)\left(-\sqrt{3}\right)$$

$$= 4\sqrt{3}a^2 - 12ap + 3\sqrt{3}p^2 - 12ap + 6\sqrt{3}p^2$$

$$= 9\sqrt{3}p^2 - 24ap + 4\sqrt{3}a^2$$

$$\frac{dV}{dp} = 0$$

$$9\sqrt{3}p^2 - 24ap + 4\sqrt{3}a^2 = 0$$

$$p = \frac{-(-24a) \pm \sqrt{(-24a)^2 - 4(9\sqrt{3})\left(4\sqrt{3}a^2\right)}}{2(9\sqrt{3})}$$

$$p = \frac{24a \pm \sqrt{144a^2}}{18\sqrt{3}} = \frac{2a}{3\sqrt{3}} \text{ or } \frac{2a}{\sqrt{3}}$$

When $p=\frac{2a}{3\sqrt{3}}$, $y=4a-2\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)=4a-\frac{4}{3}a=\frac{8}{3}a>0$

When $p=\frac{2a}{\sqrt{3}}$, $y=4a-2\sqrt{3}\left(\frac{2a}{\sqrt{3}}\right)=4a-4a=0$ (NA as $y\neq 0$)

Or

When $p=\frac{2a}{3\sqrt{3}}$, $V=\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)\left(2a-\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)\right)^2=\frac{2a}{3}\left(2a-\frac{2a}{3}\right)^2=\frac{32}{27}a^3>0$

When $p=\frac{2a}{\sqrt{3}}$, $V=\sqrt{3}\left(\frac{2a}{\sqrt{3}}\right)\left(2a-\sqrt{3}\left(\frac{2a}{\sqrt{3}}\right)\right)^2=2a\left(2a-2a\right)^2=0$ (NA as $V\neq 0$)

Therefore, $p=\frac{2a}{3\sqrt{3}}=\frac{2\sqrt{3}a}{9}$ cm

(iii)

$\frac{\mathrm{d}V}{\mathrm{d}p}=9\sqrt{3}p^2-24ap+4\sqrt{3}a^2$

$\frac{\mathrm{d}^2V}{\mathrm{d}p^2}=18\sqrt{3}p-24a$

When $p=\frac{2a}{3\sqrt{3}}=\frac{2\sqrt{3}a}{9}$, $\frac{\mathrm{d}^2V}{\mathrm{d}p^2}=18\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)-24a=12a-24a=-12a<0$ (max)

Hence, V is maximum when $p=\frac{2a}{3\sqrt{3}}=\frac{2\sqrt{3}a}{9}$

Maximum volume of the prism = $\frac{\sqrt{3}}{4}p\left(4a-2\sqrt{3}p\right)^2=\frac{32}{27}a^3$ cm^3

$\frac{3}{4}$ of the maximum volume = $\frac{3}{4}\times\frac{32}{27}\times a^2=\frac{8}{9}a^3$ cm^3

Maximum cost that the housewife has to pay = $\frac{8}{9}a^3\times 0.4=\frac{16}{45}a^3$ cents.

9

(a)

$\int\frac{e^{\frac{1}{x}}}{x^2}\,\mathrm{d}x=-\int-\frac{1}{x^2}e^{\frac{1}{x}}\,\mathrm{d}x$

$=-e^{\frac{1}{x}}+C$

(b)

$$\int \cos kx \cos(k+2)x \, dx = \frac{1}{2}\int 2\cos(k+2)x\cos kx \, dx$$
$$= \frac{1}{2}\int \left[\cos(2k+2)x + \cos 2x\right] dx$$
$$= \frac{1}{2}\left[\frac{\sin(2k+2)x}{2k+2} + \frac{\sin 2x}{2}\right] + C$$

(c)

$$\int x\tan^{-1}(3x)\, dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\, dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}\int 1 - \frac{1}{1+9x^2}\, dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}\int 1\, dx + \frac{1}{6}\int \frac{1}{9\left(\frac{1}{9}+x^2\right)}\, dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}x + \frac{1}{18}\tan^{-1}(3x) + C$$

$u = \tan^{-1}(3x)$	$\frac{dv}{dx} = x$
$\frac{du}{dx} = \frac{3}{1+9x^2}$	$v = \frac{1}{2}x^2$

Q10

(a)(i)

Using AP, $v_n = 44\ 000 + (n-1)(335) = 43665 + 335n$

(ii)

Using $v_n > 53\ 500$

$43665 + 335n > 53\ 500$

$n > 29.358$

least $n = 30$

Average speed of 53 500 km/hr was reached in March 2001.

(iii)

$T =$ 1 month = 30(24) hours = 720 hours , 3 years = 36 months

Distance = $720\left[\frac{36}{2}(2(44000)+35(335))\right]$ = 1 292 436 000 km

(b)(i)

GP : 0.44, 0.44(2.03) , $0.44(2.03)^2$, ………

longest orbital period = $0.44(2.03)^{66} = 8.67 \times 10^{19}$ days

(ii)

$$\left|S_n - 0.44(2.03)^{19}\right| < 5\times10^6$$

$$\left|\frac{0.44(2.03^n - 1)}{2.03-1} - 0.44(2.03)^{19}\right| < 5\times10^6$$

Using GC (table) , largest n = 23

OR:

$$-5\times10^6 < \frac{0.44(2.03^n - 1)}{2.03-1} - 0.44(2.03)^{19} < 5\times10^6$$

$$-5\times10^6 + 306110.3422 < \frac{0.44\left(2.03^n\right)}{1.03} < 5\times10^6 + 306110.3422$$

Since $\frac{0.44\left(2.03^n\right)}{1.03}$ is always positive, $\frac{0.44\left(2.03^n\right)}{1.03} < 5\times10^6 + 306110.3422$

Solving, $n < 23.0707$

largest n = 23

Q1

$$a_{n+1} = a_n + 3^n - n$$

$$\sum_{r=0}^{n-1}\left(a_{r+1} - a_r\right) = \sum_{r=0}^{n-1}\left(3^r - r\right)$$

$$= \sum_{r=0}^{n-1} 3^r - \sum_{r=0}^{n-1} r$$

$$\begin{aligned}
&a_1 - a_0 \\
&+a_2 - a_1 \\
&+a_3 - a_2 \\
&+ \\
&\quad\vdots \\
&+a_{n-2} - a_{n-3} = \frac{1(3^n - 1)}{3-1} - \frac{(n-1)n}{2} \\
&+a_{n-1} - a_{n-2} \\
&+a_n - a_{n-1}
\end{aligned}$$

$$a_n - a_0 = \frac{(3^n - 1)}{2} - \frac{(n-1)n}{2}$$

$$a_n = a_0 + \frac{(3^n - 1)}{2} - \frac{(n-1)n}{2}$$

$$= \frac{3}{5} - \frac{1}{2} + \frac{3^n}{2} - \frac{(n-1)n}{2}$$

$$= \frac{1}{10} + \frac{3^n}{2} - \frac{(n-1)n}{2}$$

Q2

(a)(i)

$$\ln(\cos 3x) \approx \ln\left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!}\right) = \ln\left(1 + \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)\right)$$

$$\approx \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) - \frac{\left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)^2}{2} + \frac{\left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)^3}{3}$$

$$\approx \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) - \frac{1}{2}\left(\frac{81}{4}x^4 - \frac{243}{8}x^6\right) + \frac{1}{3}\left(-\frac{729}{8}x^6\right)$$

$$= -\frac{9}{2}x^2 - \frac{27}{4}x^4 - \frac{81}{5}x^6$$

(a)(ii)

Differentiating wrt x,

$$\frac{-3\sin 3x}{\cos 3x} \approx -9x - 27x^3 - \frac{486}{5}x^5$$

$$-3\tan 3x \approx -9x - 27x^3 - \frac{486}{5}x^5$$

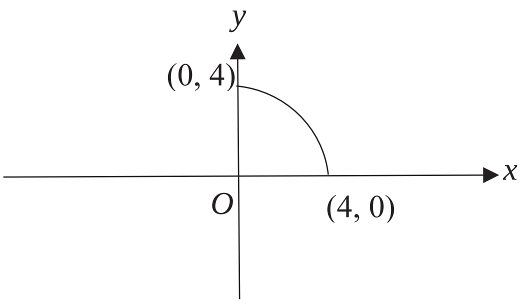
$$\tan 3x \approx 3x + 9x^3 + \frac{162}{5}x^5$$

(b)

$$\frac{e^{\tan x}}{(2+x)^2}$$
$$= e^{\tan x}(2+x)^{-2}$$
$$= \frac{1}{4}e^{\tan x}\left(1+\frac{x}{2}\right)^{-2}$$
$$\approx \frac{1}{4}e^{x}\left(1+\frac{x}{2}\right)^{-2}$$
$$\approx \frac{1}{4}\left(1+x+\frac{1}{2}x^2\right)\left(1-x+\frac{3}{4}x^2\right)$$
$$= \frac{1}{4}\left(1+\frac{1}{4}x^2\right)$$

Q3

(i)



(ii)

$x = 4\sin 2t$ $\quad\quad$ $y = 4\cos 2t$

$\frac{dx}{dt} = 8\cos 2t$ $\quad\quad$ $\frac{dy}{dt} = -8\sin 2t$,

$\frac{dy}{dx} = -\tan 2t$

(a) For tangents parallel to y-axis ($\frac{dy}{dx}$ is undefined): $2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}$

(b) For tangent parallel to x-axis ($\frac{dy}{dx} = 0$): $2t = 0 \Rightarrow t = 0$

(iii)

Given that gradient of tangent = $-\dfrac{1}{\sqrt{3}}$

$-\tan 2t = -\dfrac{1}{\sqrt{3}}$,

$2t = \dfrac{\pi}{6}$

$\therefore \quad t = \dfrac{\pi}{12}$

When $t = \dfrac{\pi}{12}$, $x = 4\sin 2\left(\dfrac{\pi}{12}\right) = 2$, $y = 4\cos 2\left(\dfrac{\pi}{12}\right) = 2\sqrt{3}$

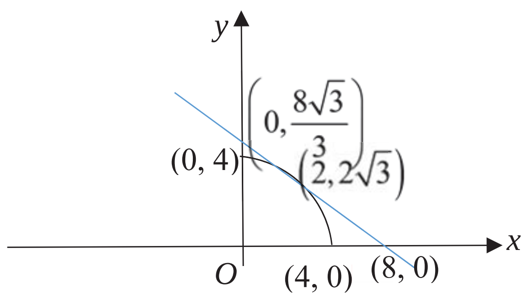
Equation of tangent at $P(2,\ 2\sqrt{3})$:

$y - 2\sqrt{3} = -\dfrac{1}{\sqrt{3}}(x-2)$

$y = -\dfrac{\sqrt{3}}{3}x + \dfrac{8}{3}\sqrt{3} = \dfrac{\sqrt{3}}{3}(8-x)$, where $a = \sqrt{3}$, $b = 8$.

(iv)

When $x = 0$, $y = \dfrac{8\sqrt{3}}{3}$. When $y = 0$, $x = 8$



Required area

= Area of the triangle – Area of the quadrant of the circle

$= \int_0^8 \dfrac{\sqrt{3}}{3}(8-x)\ \mathrm{d}x - \dfrac{1}{4}\pi(4)^2$

$= \left(\dfrac{1}{2}\times 8 \times \dfrac{8\sqrt{3}}{3}\right) - \dfrac{1}{4}\pi(4)^2 = \left(\dfrac{32\sqrt{3}}{3} - 4\pi\right)$ units ^{2}

Q4

$\dfrac{\mathrm{d}}{\mathrm{d}u}\left[u^2\dfrac{\mathrm{d}x}{\mathrm{d}u}\right] = u^2\dfrac{\mathrm{d}^2x}{\mathrm{d}u^2} + 2u\dfrac{\mathrm{d}x}{\mathrm{d}u}$

$2u^2\dfrac{\mathrm{d}^2x}{\mathrm{d}u^2} + 4u\dfrac{\mathrm{d}x}{\mathrm{d}u} = 15u + 12$ becomes $2\dfrac{\mathrm{d}}{\mathrm{d}u}\left[u^2\dfrac{\mathrm{d}x}{\mathrm{d}u}\right] = 15u + 12$

Thus $u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\displaystyle\int 15u + 12\ \mathrm{d}u$

$u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\left[\dfrac{15u^2}{2} + 12u\right] + C$

$x = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}u} = 1$ when $u = 1$

$C = -\dfrac{35}{4}$

$u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\left[\dfrac{15u^2}{2} + 12u\right] - \dfrac{35}{4}$,

$$\frac{dx}{du}=\frac{1}{2}\left[\frac{15}{2}+\frac{12}{u}\right]-\frac{35}{4u^2}$$

$$x=\int \frac{1}{2}\left[\frac{15}{2}+\frac{12}{u}\right]-\frac{35}{4u^2}\ du$$

$$=\ \frac{15}{4}u+6\ln|u|+\frac{35}{4u}+D$$

$x=0$ when $u=1$

$\therefore\ D=-\frac{25}{2}$

$$x=\frac{15}{4}u+6\ln|u|+\frac{35}{4u}-\frac{25}{2}$$

Q5

(i)

Method 1

$$\begin{pmatrix}8\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}1\\2\\0\end{pmatrix}=8-8+0=0$$

$$\begin{pmatrix}8\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}2\\3\\-4\end{pmatrix}=16-12-4=0$$

Since direction vector of l is perpendicular to two vectors parallel to p, l is perpendicular to p.

Method 2

$$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-8\\4\\-1\end{pmatrix}=-\begin{pmatrix}8\\-4\\1\end{pmatrix}$$ which is parallel to l

l is perpendicular to p.

$$\begin{pmatrix}1\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-10\\0\\4\end{pmatrix}+t\begin{pmatrix}8\\-4\\1\end{pmatrix}$$

$x:\quad 1+\lambda+2\mu=-10+8t\Rightarrow\lambda+2\mu-8t=-11\quad (1)$

$y:\quad 2\lambda+3\mu=-4t\Rightarrow 2\lambda+3\mu+4t=0\quad (2)$

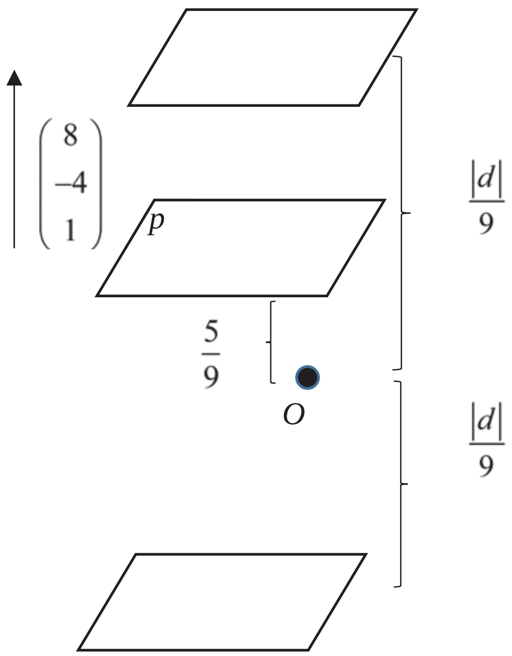
$z:\quad -3-4\mu=4+t\Rightarrow -4\mu-t=7\quad (3)$

$\lambda=1,\quad \mu=-2,\quad t=1$

(ii)

<u>Method 1</u>

Equation of p is $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-3\end{pmatrix}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=5$



Let the equations of the required planes be $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=d$

Given distance between p and planes = 2

$$\frac{|d-5|}{9}=2$$

$$|d-5|=18$$

$d-5=18$ or $d-5=-18$

$d=23$ or $d=-13$

Cartesian equations of the required planes are

$8x-4y+z=23$ and $8x-4y+z=-13$

<u>Method 2</u>

Equation of p is $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-3\end{pmatrix}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=5$

$$\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}$$

Equations of the required planes are

$\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}+2$ and $\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}-2$

$\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=23$ and $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=-13$

Cartesian equations of the required planes are

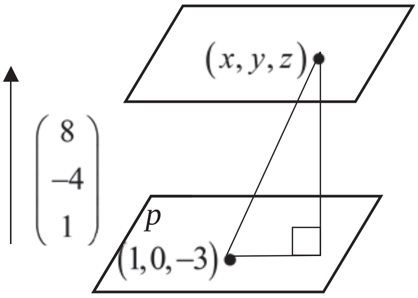
$8x-4y+z=23$ and $8x-4y+z=-13$

Method 3

Let a point on the required planes be (x, y, z) .

Consider point $(1, 0, -3)$ on p.

Given distance between p and planes = 2



$$\frac{1}{9}\left|\left[\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}1\\0\\-3\end{pmatrix}\right]\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}\right|=2$$

$$\frac{1}{9}\left|\begin{pmatrix}x-1\\y\\z+3\end{pmatrix}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}\right|=2$$

$$|8x-8-4y+z+3|=18$$

$$|8x-4y+z-5|=18$$

$8x-4y+z-5=18$ or $8x-4y+z-5=-18$

Cartesian equations of the required planes are

$8x-4y+z=23$ and $8x-4y+z=-13$

Q6

(i)

Whether a box contains voucher is independent of any other boxes.

The probability of a box containing a voucher is constant.

(ii)

Let X be the number of vouchers obtained out of 9 boxes of Brand A cereal

$$X \sim \mathrm{B}(9, 0.35)$$

$$\mathrm{P}(X \le 3) = 0.60889 \approx 0.609$$

(iii)

Required probability = $\mathrm{P}(X = 7) \times 0.35 = 0.0034251 \approx 0.00343$

Alternative

Required probability = ${}^{9}C_{7}(0.35)^{8}(0.65)^{2} = 0.0034251 \approx 0.00343$

Let W be the number of vouchers obtained out of 10 boxes of Brand B cereal

$$W \sim \mathrm{B}(10, p)$$

$$\mathrm{P}(W \le 1) = 0.4845$$

$$\binom{10}{0}p^{0}(1-p)^{10}+\binom{10}{1}p^{1}(1-p)^{9}=0.4845$$

$$(1-p)^{10}+10p(1-p)^{9}=0.4845$$

$$p = 0.16667 \approx 0.167$$

Q7

(i)
Central Limit Theorem states that sample means will follow a normal distribution approximately when sample size is big enough.
(ii)
$\sum(x-390)=120$ and $\sum(x-390)^2=3100$

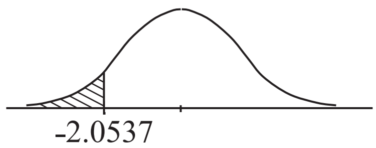
$\bar{x}=\frac{120}{50}+390=392.4$

$s^2=\frac{1}{50-1}\left[3100-\frac{120^2}{50}\right]=57.3877551\approx 57.388$

$H_0: \mu=m$ vs $H_1: \mu<m$

Since n = 50 is large, by Central Limit Theorem, $\overline{X} \sim N(m, \frac{57.388}{50})$ approximately

Level of significance: 2%
Critical region: Reject H_0 when $z<-2.0537$



Consider $\frac{392.4-m}{\sqrt{\frac{57.388}{50}}}<-2.0537$

$m>394.6$ (5 s.f.)
The least possible value of m is 395grams.

Q8
(a)

Case 1: Last digit is 5

{2,4,6,8}					{5}

Number of different 6-digit numbers = $4\times 4! = 96$

Case 2: Last digit is 0

{2,4,5,6,8}					{0}

Number of different 6-digit numbers = $5! = 120$

$\therefore$ Required number of different 6-digit numbers = 96 + 120 = 216

(b)
Case 1: All letters are different.
Number of 4-letter code words = ${}^7C_4\times 4!=840$
Case 2: One pair of repeated letters.
Number of 4-letter code words = ${}^2C_1\times {}^6C_2\times\frac{4!}{2!}=360$

Case 3: Two pairs of repeated letters.
Number of 4-letter code words = $\frac{4!}{2!2!}=6$

$\therefore$ Total number of 4-letter code words = 840 + 360 + 6 = 1206

Q9
(i)
Let A be the mass of a Grade A strawberry

$$A \sim \mathrm{N}\left(18, 3^2\right)$$

$$\mathrm{P}(17 < A < 20) = 0.37806 \approx 0.378$$

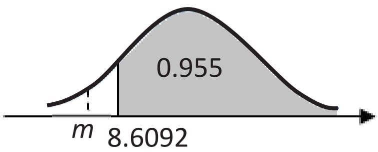
(ii)
Let B be the mass of a Grade B strawberry

$$B \sim \mathrm{N}\left(12, 2^2\right)$$

$$\mathrm{P}(B > m) \geq 0.955$$

$$m \leq 8.6092$$

Greatest value of $m = 8.60$



(iii)
Let $X = (A_1 + \ldots + A_{12}) - (B_1 + \ldots + B_{15})$

$$\mathrm{E}(X) = 12(18) - 15(12) = 36$$

$$\mathrm{Var}(X) = 12\left(3^2\right) + 15\left(2^2\right) = 168$$

$$X \sim \mathrm{N}(36, 168)$$

$$\mathrm{P}(X > 0) = 0.99726 \approx 0.997$$

(iv)
The masses of strawberries are independent of each other.

Q10
(i)

	$4n$-4 Cards	3 Cards	1 Card
Score (+)	**1**	**2**	**4**
1	2	3	5
2	3	4	6
4	5	6	Nil

$$\mathrm{P}(W = 10)$$

$$= \mathrm{P}(\text{Score} > 4)$$

$$= \mathrm{P}(1,4) + \mathrm{P}(4,1) + \mathrm{P}(2,4) + \mathrm{P}(4,2)$$

$$= \left(\frac{4n-4}{4n}\right)\left(\frac{1}{4n-1}\right) \times 2 + \left(\frac{3}{4n}\right)\left(\frac{1}{4n-1}\right) \times 2$$

$$= \frac{8n - 8 + 6}{4n(4n-1)}$$

$$= \frac{1}{2n}$$

(ii)

$$\frac{1}{2n}=\frac{1}{8}$$
$$n=4$$

$$\begin{aligned}&\mathrm{P}(W=-2)\\&=\mathrm{P}(\text{Score}<4)\\&=\mathrm{P}(1,1)+\mathrm{P}(1,2)+\mathrm{P}(2,1)\\&=\left(\frac{12}{16}\right)\left(\frac{11}{15}\right)+\left(\frac{12}{16}\right)\left(\frac{3}{15}\right)\times 2\\&=\frac{17}{20}\end{aligned}$$

$$\begin{aligned}&\mathrm{P}(W=0)\\&=\mathrm{P}(\text{Score}=4)\\&=\mathrm{P}(2,2)\\&=\left(\frac{3}{16}\right)\left(\frac{2}{15}\right)\\&=\frac{1}{40}\end{aligned}$$

Hence, the probability distribution of W is

W	-2	0	10
$\mathrm{P}(W=w)$	$\frac{17}{20}$	$\frac{1}{40}$	$\frac{1}{8}$

(iii)

$$\mathrm{E}(W)=(-2)\left(\frac{17}{20}\right)+(0)\left(\frac{1}{40}\right)+(10)\left(\frac{1}{8}\right)$$
$$=-\frac{9}{20}$$

$$\mathrm{E}(W^2)=(-2)^2\left(\frac{17}{20}\right)+(0)^2\left(\frac{1}{40}\right)+(10)^2\left(\frac{1}{8}\right)$$
$$=\frac{159}{10}$$

$$\mathrm{Var}(W)=\mathrm{E}(W^2)-\left[\mathrm{E}(W)\right]^2$$
$$=\frac{159}{10}-\left(-\frac{9}{20}\right)^2$$
$$=\frac{6279}{400}\quad\text{or } 15.6975 \text{ (exact)}$$

Since $\mathrm{E}(W)=-\frac{9}{20}<0$, it is expected that she will lose money. Hence, Kathryn should not play the game.

(iv)

Since $n = 50$ is large, by Central Limit Theorem, $\overline{W} \sim \mathrm{N}\left(-\frac{9}{20}, \frac{15.6975}{50}\right)$ approximately

$\mathrm{P}\left(\overline{W} \le 1\right) = 0.99517 \approx 0.995$

Q11

(i)

Given $\mathrm{P}(A|B) = \frac{7}{10}$

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{7}{10}$$

$\therefore\ \mathrm{P}(B) = \frac{10}{7}\mathrm{P}(A \cap B)$ -----(1)

Given $\mathrm{P}(B|A) = \frac{4}{15}$

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{4}{15}$$

$\therefore\ \mathrm{P}(A) = \frac{15}{4}\mathrm{P}(A \cap B)$ -----(2)

Given that $\mathrm{P}(A \cup B) = \frac{3}{5}$

$\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{3}{5}$ -----(3)

Substitute (1) and (2) into (3):

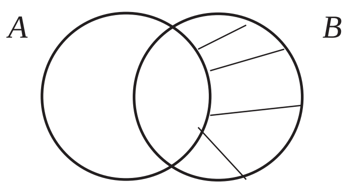
$$\frac{15}{4}\mathrm{P}(A \cap B) + \frac{10}{7}\mathrm{P}(A \cap B) - \mathrm{P}(A \cap B) = \frac{3}{5}$$

$\therefore\ \mathrm{P}(A \cap B) = \frac{28}{195}$

(ii)

$\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$$= \frac{10}{7}\left(\frac{28}{195}\right) - \frac{28}{195} = \frac{4}{65}$$



Alternative Method 1:

$\mathrm{P}(A' \cap B) = \mathrm{P}(A \cup B) - \mathrm{P}(A) = \frac{3}{5} - \left(\frac{15}{4} \times \frac{28}{195}\right) = \frac{4}{65}$

Alternative Method 2:

$\mathrm{P}(A'|B) = 1 - \mathrm{P}(A|B) = 1 - \frac{7}{10} = \frac{3}{10}$

$\therefore\ \mathrm{P}(A' \cap B) = \mathrm{P}(A'|B) \times \mathrm{P}(B) = \frac{3}{10} \times \left(\frac{10}{7} \times \frac{28}{195}\right) = \frac{4}{65}$

(iii)

$\mathrm{P}(A)=\frac{15}{4}\left(\frac{28}{195}\right)=\frac{7}{13}$

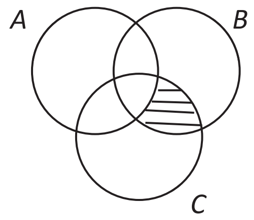
Since events A and C are independent,

$\mathrm{P}(A'\cap C)=\mathrm{P}(A')\times\mathrm{P}(C)=\left(1-\frac{7}{13}\right)\times\frac{3}{10}=\frac{9}{65}$

(iv)

$\mathrm{P}(A'\cap B\cap C)\le\mathrm{P}(A'\cap C)$

$\therefore\ 0\le\mathrm{P}(A'\cap B\cap C)\le\frac{9}{65}$

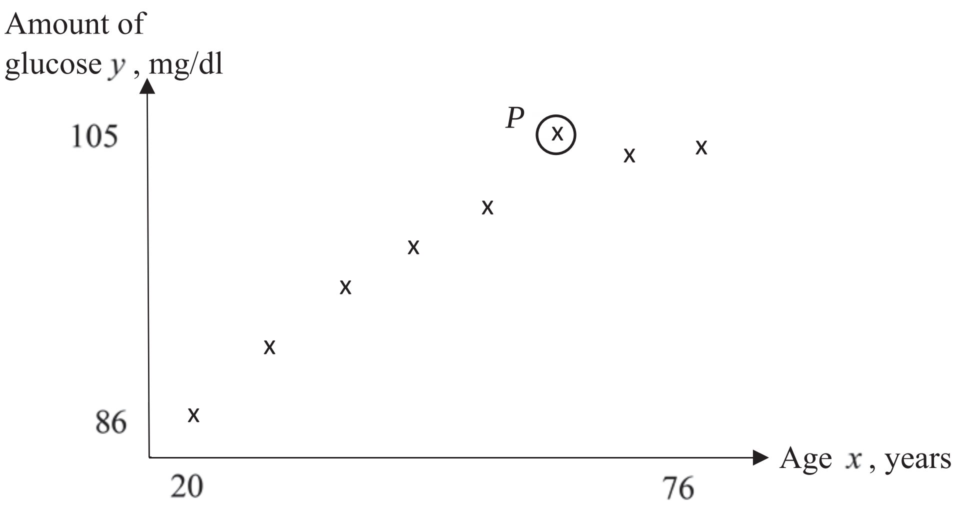


Q12

(i)

$r=0.954$. Since the r value is close to 1, there is a strong positive linear correlation between x and y. A linear model may be appropriate.

(ii) and (iii)



(iv)

From the scatter diagram, excluding P, it can be observed that as x increases, y also increases but at a decreasing rate, hence a linear model $y=ax+b$ may not be appropriate.

(v)

From GC,

$r=0.998$

$y=44.281+13.972\ln x$

$c=44.3 \qquad d=14.0$

(vi)

When $x=60$, $y=44.281+13.972\ln 60=101.5$ (nearest 0.5)

$r=0.998$ is close to 1. $x=60$ lies within the given data range, hence interpolation is being done. The linear model $y=a+b\ln x$ still holds, hence, the estimate is reliable.

(vii)

From GC, mean of $\ln x$ = 3.7864

$\bar{y} = 43.942 + 14.079(3.7864) = 97.251$

$\bar{y} = 97.251 = \frac{1}{8}(86.0 + 90.5 + 94.5 + 97.5 + 100 + 103.5 + 104 + y)$

$y = 102.0$ (nearest 0.5)