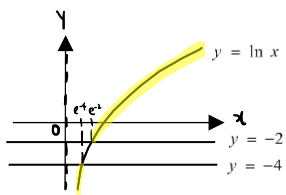
DHS 2022 Year 6 H2 Math Prelim Exam P1 solutions

Qn	Suggested Solution
1(a)	$-x^2+6x-14=-\left(x^2-6x+14\right)$ $=-\left[\left(x-3\right)^2+5\right]$ $=-\left(x-3\right)^2-5<0$ for all real values of x $\left(\because \left(x-3\right)^2\geq 0\right)$ **Alternative** Discriminant of $-x^2+6x-14=6^2-4\left(-1\right)\left(-14\right)$ $=36-56$ $=-20<0$ Since coeff of $x^2<0\Rightarrow -x^2+6x-14<0$ for all real values of x
(b)	$\dfrac{x-5}{x-2}\geq\dfrac{3}{4-x}$ $\dfrac{\left(x-5\right)\left(4-x\right)-3\left(x-2\right)}{\left(x-2\right)\left(4-x\right)}\geq 0$ $\dfrac{-x^2+6x-14}{\left(x-2\right)\left(4-x\right)}\geq 0$ Since $-x^2+6x-14<0$ for all real x, $\left(x-2\right)\left(4-x\right)<0$ $x<2$ or $x>4$
(c)	$\dfrac{x-5}{x-2}>\dfrac{3}{4-x}$, Replace x with $-\ln x$. $\dfrac{\ln x+5}{\ln x+2}>\dfrac{3}{4+\ln x}$. $-\ln x<2$ or $-\ln x>4$ $\ln x>-2$ or $\ln x<-4$ $x>e^{-2}$ or $0<x<e^{-4}$ 

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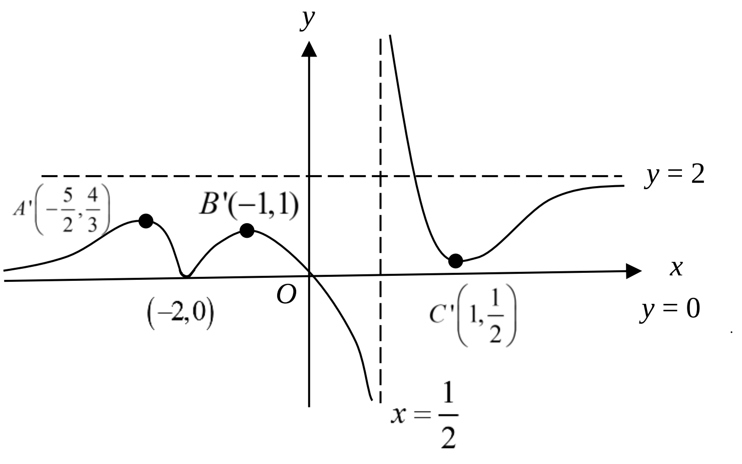
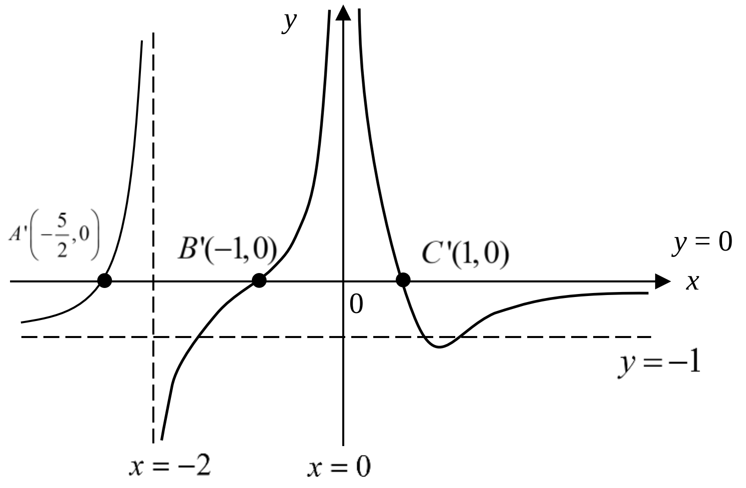
2(a)	
(b)	

Qn	Suggested Solution
3(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\sin^2 2x} = 4\mathrm{e}^{\sin^2 2x}\sin 2x\cos 2x = 2\mathrm{e}^{\sin^2 2x}\sin 4x$
(b)	$\int \frac{\mathrm{e}^{\sin^2 2x}\sin 4x}{\sqrt{1+\mathrm{e}^{\sin^2 2x}}}\,\mathrm{d}x$ $=\frac{1}{2}\int\left(2\mathrm{e}^{\sin^2 2x}\sin 4x\right)\left(1+\mathrm{e}^{\sin^2 2x}\right)^{-\frac{1}{2}}\,\mathrm{d}x$ $=\left(1+\mathrm{e}^{\sin^2 2x}\right)^{\frac{1}{2}}+c$ $=\sqrt{1+\mathrm{e}^{\sin^2 2x}}+c$
(c)	$\int_0^{\frac{\pi}{4}}\left(\mathrm{e}^{\sin^2 2x}\sin 4x\right)\cos^2 2x\,\mathrm{d}x$ $=\left[\frac{1}{2}\mathrm{e}^{\sin^2 2x}\cos^2 2x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{1}{2}\mathrm{e}^{\sin^2 2x}\left(-4\cos 2x\sin 2x\right)\,\mathrm{d}x$ $=-\frac{1}{2}+\int_0^{\frac{\pi}{4}}\mathrm{e}^{\sin^2 2x}\sin 4x\,\mathrm{d}x$ $=-\frac{1}{2}+\left[\frac{1}{2}\mathrm{e}^{\sin^2 2x}\right]_0^{\frac{\pi}{4}}$ $=-\frac{1}{2}+\frac{1}{2}\mathrm{e}-\frac{1}{2}$ $=\frac{1}{2}\mathrm{e}-1$

Qn	Suggested Solution
4	**Method 1** $(\cos\theta+\mathrm{i}\sin\theta)^3+(\cos\theta+\mathrm{i}\sin\theta)^5=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $(\cos3\theta+\mathrm{i}\sin3\theta)+(\cos5\theta+\mathrm{i}\sin5\theta)=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $(\cos3\theta+\cos5\theta)+\mathrm{i}(\sin3\theta+\sin5\theta)=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $2\cos4\theta\cos\theta+2\mathrm{i}\sin4\theta\cos\theta=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $2\cos\theta[\cos4\theta+\mathrm{i}\sin4\theta]=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $[2\cos\theta]\mathrm{e}^{\mathrm{i}(4\theta)}=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $\left\|(2\cos\theta)\mathrm{e}^{\mathrm{i}(4\theta)}\right\|=\left\|\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}\right\|=1$ $\|2\cos\theta\|=1 \;\Rightarrow\; \theta=\pm\frac{\pi}{3}$ Comparing the argument of 4θ with $-\frac{2\pi}{3}$, only $\theta=\frac{\pi}{3}$ is valid. **Method 2** $(\cos\theta+\mathrm{i}\sin\theta)^3+(\cos\theta+\mathrm{i}\sin\theta)^5=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ $(\cos3\theta+\cos5\theta)+\mathrm{i}(\sin3\theta+\sin5\theta)=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ $2\cos4\theta\cos\theta+2\mathrm{i}\sin4\theta\cos\theta=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ Comparing real and imaginary parts, $2\cos4\theta\cos\theta=-\frac{1}{2}$ (1) $2\sin4\theta\cos\theta=-\frac{\sqrt{3}}{2}$ (2) $\frac{(2)}{(1)}$, $\tan4\theta=\sqrt{3}$ $4\theta=-\frac{5\pi}{3},-\frac{2\pi}{3},\frac{\pi}{3},\frac{4\pi}{3}$ $\theta=-\frac{5\pi}{12},-\frac{\pi}{6},\frac{\pi}{12},\frac{\pi}{3}$ Only $\theta=\frac{\pi}{3}$ satisfies (1) and (2).

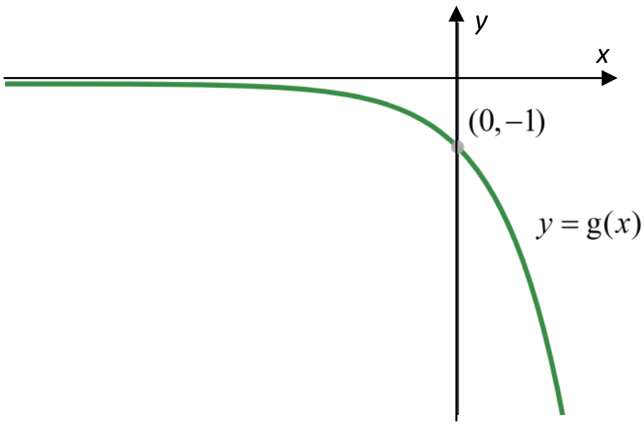
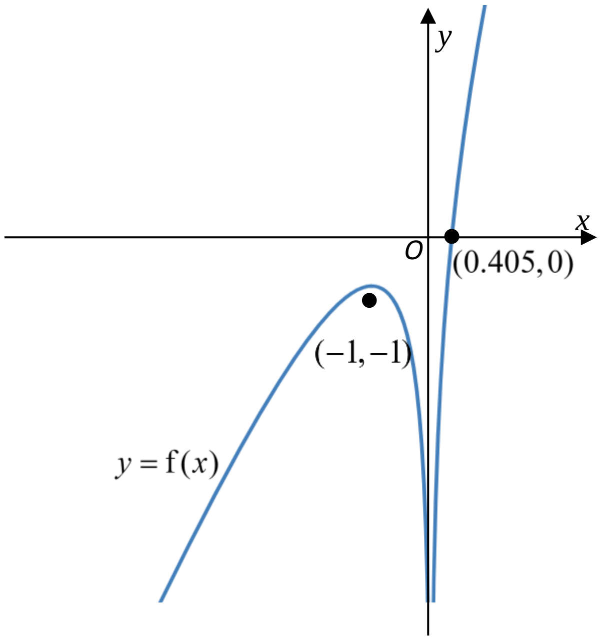
Qn	Suggested Solution
5(a)	$y^2 = 2\sin x + 2xy$ ... (1) Differentiate with respect to x, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x + 2\left(y + x\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$ $2y\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x + 2y$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos x + y}{y - x}$ (shown)
(b)	For stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos x + y}{y - x} = 0$ $\cos x + y = 0$ $y = -\cos x$ Substitute $y = -\cos x$ into (1) $(-\cos x)^2 = 2\sin x + 2x(-\cos x)$ $\cos^2 x - 2\sin x + 2x\cos x = 0$ From GC : $x = 0.87394$ or 4.4877 $y = -\cos(0.87394)$ or $-\cos(4.4877)$ $= -0.64181$ or 0.22280 Coordinates : $P(0.874, -0.642)$ & $Q(4.49, 0.223)$ $(y - x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x + y$ $(y - x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x} - 1\right) = -\sin x + \dfrac{\mathrm{d}y}{\mathrm{d}x}$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $(y - x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\sin x$ At $x = 0.87394$ and $y = -0.64181$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.50593 > 0$ $\therefore (0.874, -0.642)$ is a minimum point. At $x = 4.4877$ and $y = 0.22280$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -0.22858 < 0$ $\therefore (4.49, 0.223)$ is a maximum point.

Qn	Suggested Solution
6(a)	Method 1 $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sin\left[\ln(1+x)\right]}{1+x}$ $\therefore (1+x)\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin\left[\ln(1+x)\right]$ Differentiating with respect to x, $(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\cos\left[\ln(1+x)\right]}{1+x}=-\frac{y}{1+x}$ $\therefore (1+x)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}+y=0$ (shown) Method 2 $\cos^{-1}y=\ln(1+x)$ Differentiating with respect to x, $-\frac{1}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{1}{1+x}$ $\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{1-y^2}$ Differentiating with respect to x, $\left(1+x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\frac{1}{2}\left(-2y\right)}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{y}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\left(1+x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{1+x}\quad\because \frac{1}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\frac{1}{1+x}$ $\left(1+x\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-y$ $\left(1+x\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}+y$ (shown)
	Differentiating with respect to x, $(1+x)^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\therefore (1+x)^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ Differentiating again with respect to x, $(1+x)^2\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+2(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $\therefore (1+x)^2\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+5(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+5\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ When $x=0,\ y=1,\frac{\mathrm{d}y}{\mathrm{d}x}=0,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1,\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=3,\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-10$ $\therefore y=1-\frac{1}{2!}x^2+\frac{3}{3!}x^3-\frac{10}{4!}x^4+\dots$ $\quad=1-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{12}x^4+\dots$ (shown)

(b)	$\sin\left[\ln(1+x)\right]=-(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $=-(1+x)\left(-x+\frac{3}{2}x^2-\frac{5}{3}x^3+\ldots\right)$ $=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$
(c)	$\therefore$ Equation of tangent to curve at $x=0$ is $y=x$.

	Suggested Solution
7(a)(i)	$\displaystyle\sum_{r=0}^{n}\frac{(x+3)^r}{4^{r+1}}=\frac{1}{4}\sum_{r=0}^{n}\left(\frac{x+3}{4}\right)^r$ $\displaystyle=\frac{1}{4}\sum_{r=0}^{n}\left[\left(\frac{x+3}{4}\right)^0+\left(\frac{x+3}{4}\right)^1+\ldots+\left(\frac{x+3}{4}\right)^n\right]$ $\displaystyle=\frac{1}{4}\left[\frac{1-\left(\frac{x+3}{4}\right)^{n+1}}{1-\frac{x+3}{4}}\right]$ $\displaystyle=\frac{1}{4}\left[\frac{1-\left(\frac{x+3}{4}\right)^{n+1}}{\frac{1-x}{4}}\right]$ $\displaystyle=\frac{1}{1-x}\left[1-\left(\tfrac{x+3}{4}\right)^{n+1}\right]$
(ii)	Common ratio, r of G.P. = $\dfrac{x+3}{4}$ When $x=-5$, $r=\dfrac{-5+3}{4}=-\dfrac{1}{2}$. Since $\lvert r\rvert=\dfrac{1}{2}<1$, the G.P. converges. Hence, the series $\displaystyle\sum_{r=0}^{n}\frac{(x+3)^r}{4^{r+1}}$ converges. $\displaystyle\lim_{n\to\infty}\sum_{r=0}^{n}\frac{(-5+3)^r}{4^{r+1}}=\lim_{n\to\infty}\frac{1}{1-(-5)}\left[1-\left(\tfrac{-5+3}{4}\right)^{n+1}\right]$ $\displaystyle=\lim_{n\to\infty}\frac{1}{6}\left[1-\left(-\tfrac{1}{2}\right)^{n+1}\right]$ $\displaystyle=\frac{1}{6}$

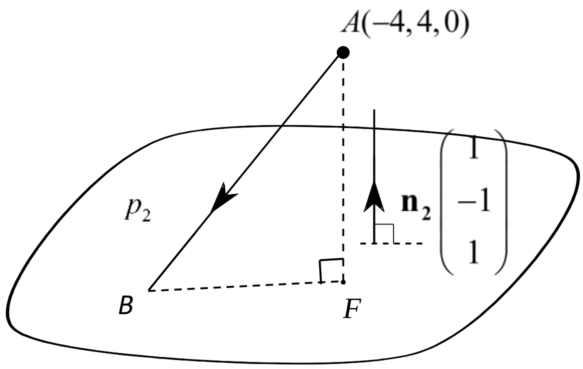
(b)(i)	$\sum_{r=6}^{2k} r(3r-2)=\sum_{r=6}^{2k}\left(3r^2-2r\right)$ $=3\left[\sum_{r=1}^{2k} r^2-\sum_{r=1}^{5} r^2\right]-2\sum_{r=6}^{2k} r$ $=3\left[\frac{2k}{6}(2k+1)(4k+1)-\frac{5}{6}(6)(11)\right]$ $-2\left(\frac{2k-6+1}{2}(6+2k)\right)$ $=k(2k+1)(4k+1)-165-2(2k-5)(3+k)$ $=k(2k+1)(4k+1)-2(2k-5)(k+3)-165$
(ii)	<u>**Method 1**</u> Replace r with r + 4 in $\sum_{r=10}^{66}(r-4)(3r-14)$. $\sum_{r+4=10}^{r+4=66}\left((r+4)-4\right)\left(3(r+4)-14\right)=\sum_{r=6}^{r=62} r(3r-2)$ Compared with (b)(i), observe that k = 31. $\sum_{r=10}^{66}(r-4)(3r-14)$ $=\sum_{r=6}^{r=62} r(3r-2)$ $=31\left(2(31)+1\right)\left(4(31)+1\right)-2\left(2(31)-5\right)\left((31)+3\right)-165$ $=240084$ <u>**Method 2**</u> Replace r with r – 4 in $\sum_{r=6}^{2k} r(3r-2)$. $\sum_{r-4=6}^{r-4=2k}(r-4)\left(3(r-4)-2\right)=k(2k+1)(4k+1)-2(2k-5)(k+3)-165$ $\sum_{r=10}^{2k+4}(r-4)(3r-14)=k(2k+1)(4k+1)-2(2k-5)(k+3)-165$ Observe that $2k+4=66\Rightarrow k=31$. $\sum_{r=10}^{66}(r-4)(3r-14)$ $=31\left(2(31)+1\right)\left(4(31)+1\right)-2\left(2(31)-5\right)\left((31)+3\right)-165$ $=240084$

Qn	Suggested Solution
8(a)	
(b)	For composite function fg to exists, $R_g \subseteq D_f$. Since $(-\infty,0) \subseteq (-\infty,0)\cup(0,\infty)$, fg exists. $\mathbb{R} \xrightarrow{g} (-\infty,0) \xrightarrow{f} (-\infty,-1]$. $R_{fg} = (-\infty,-1]$.

(c)	Let m = g(x), $\mathrm{fg}(x) = 14x + 1 - 2\mathrm{e}^{7x}$ $\mathrm{f}(m) = 14x + 1 - 2\mathrm{e}^{7x}$ $\ln(m^2) + 2(m) + 1 = 14x + 1 - 2\mathrm{e}^{7x}$ $\ln(m^2) + 2(m) = 14x - 2\mathrm{e}^{7x}$ Guess $2m = -2\mathrm{e}^{7x}$ $m = -\mathrm{e}^{7x}$ Verify $\ln(m^2)$ $= \ln\left[(-e^{7x})^2\right]$ $= \ln\left(\mathrm{e}^{14x}\right)$ $= 14x$ $\therefore\ \mathrm{g}(x) = -\mathrm{e}^{7x}$.
(d)	From the graph, the least value of k is 0. $\mathrm{f}(1) = \ln 1^2 + 2 + 1 = 0 + 2 + 1 = 3$ (verified) Gradient of $y = \mathrm{f}^{-1}(x)$ at (x = 3) = 1/(Gradient of $y = \mathrm{f}(x)$) at (x = 1) $= \dfrac{1}{\mathrm{f}'(1)} = 0.250$

Qn	Suggested Solutions
9(a)	To find eqn of the line, $\begin{cases} \alpha = \dfrac{x-2}{2} \Rightarrow x = 2+2\alpha \\ \alpha = \dfrac{y-4}{3} \Rightarrow y = 4+3\alpha \\ \quad z = 6 \end{cases} \Rightarrow \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R}$ Normal of plane p_1 $= \left(\begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right) \times \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 18 \\ -12 \\ -18 \end{pmatrix} = 6 \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$ Let $\mathbf{n_1} = \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$ Normal of p_2, $\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ **<u>Method 1</u>** For direction vector of line of intersection, $\mathbf{n_2} \times \mathbf{n_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$ Hence the line of intersection is parallel to $\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$. (shown) Vector equation of the line of intersection is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}, \beta \in \mathbb{R}$ **<u>Method 2</u>** $p_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = -12-8 = -20 \Rightarrow 3x-2y-3z = -20$ $p_2 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 2-4+6 = 4 \Rightarrow x-y+z = 4$

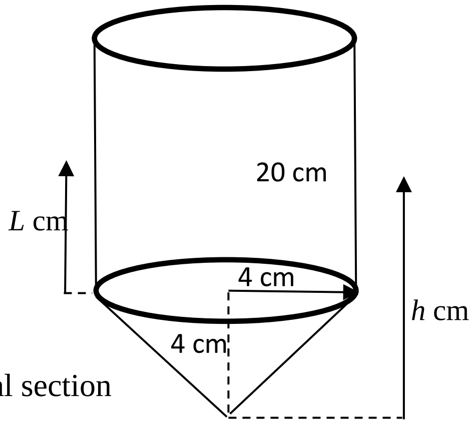
	Solve p_1 and p_2 using GC: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -28+5z \\ -32+6z \\ z \end{pmatrix} = \begin{pmatrix} -28 \\ -32 \\ 0 \end{pmatrix} + z\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$ Let $\beta = z$, we obtain the equation of the line of intersection as $\mathbf{r} = \begin{pmatrix} -28 \\ -32 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R}$. Hence the line of intersection is parallel to $\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$. (shown)
(b)	Acute angle between p_1 and p_2 $= \cos^{-1}\left\|\dfrac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}}{\sqrt{3}\sqrt{22}}\right\| = \cos^{-1}\left\|\dfrac{2}{\sqrt{3}\sqrt{22}}\right\| = 75.7^o$
(c)	From Q9(a) **Method 2**, $p_2 : \mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4$ Let the foot of perpendicular from *A* to p_2 be *F*. $l_{AF} : \mathbf{r} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ \gamma \in \mathbb{R}$ Since *F* lies on l_{AF}, $\overrightarrow{OF} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ for some values of γ

	Since F also lies on p_2,  $\begin{pmatrix} -4+\gamma \\ 4-\gamma \\ \gamma \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4$ $-4+\gamma-4+\gamma+\gamma=4$ $3\gamma=12 \Rightarrow \gamma=4$ $\overrightarrow{OF} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$
(d)	Since all three planes have a common line of intersection, the point (2, 4, 6) must also lie on p_3. Substitute (2, 4, 6) into $ax+3y+2z=b$. $\begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = b \Rightarrow 2a+24=b$ and the normal of p_3 must be perpendicular to the line of intersection, $\begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix} = 0 \Rightarrow 5a+18+2=0 \Rightarrow a=-4$ Therefore $a=-4$, $b=16$.

Qn	Suggested Solution
10(a)	$\frac{\mathrm{d}v}{\mathrm{d}t}=5-0.2v^2$ $\int \frac{1}{5-0.2v^2}\,\mathrm{d}v=\int \mathrm{d}t$ $5\int \frac{1}{25-v^2}\,\mathrm{d}v=\int \mathrm{d}t$ $5\left[\frac{1}{2(5)}\ln\left\|\frac{5+v}{5-v}\right\|\right]+C=t$ Since $\frac{\mathrm{d}v}{\mathrm{d}t}=5-0.2v^2>0$ $\Rightarrow 0\le v<5 \quad \text{as } v\ge 0$ $5\left[\frac{1}{2(5)}\ln\left(\frac{5+v}{5-v}\right)\right]+C=t$, where C is an arbitrary constant $\therefore\ t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)+C$ Since at $t=0$, $v=0$; $0=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)+C$ $\Rightarrow C=0$ $\therefore t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)$
(b)	Since $\frac{\mathrm{d}v}{\mathrm{d}t}=5-0.2v^2$, The velocity of the object is increasing at a decreasing rate. **<u>Method 1</u>** $t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)\Rightarrow\ v=\frac{5(\mathrm{e}^{2t}-1)}{\mathrm{e}^{2t}+1}=\frac{5(1-\mathrm{e}^{-2t})}{1+\mathrm{e}^{-2t}}$ As $t\to\infty,\ v\to 5$. In the long run, the velocity of the object will travel at 5 m/s. **<u>Method 2</u>** $t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)\Rightarrow \mathrm{e}^{2t}=\frac{5+v}{5-v}$ As $t\to\infty\Rightarrow 5-v\to 0\Rightarrow v\to 5$

(c)	$\int_0^m v \, \mathrm{d}t$ $= \int_0^m \frac{5(\mathrm{e}^{2t}-1)}{(\mathrm{e}^{2t}+1)} \, \mathrm{d}t$ $= 5\left(\int_0^m \frac{(\mathrm{e}^{2t}+1)-2}{\mathrm{e}^{2t}+1} \, \mathrm{d}t\right)$ $= 5\left(\int_0^m 1-\frac{2}{\mathrm{e}^{2t}+1} \, \mathrm{d}t\right)$ $= 5\left(\int_0^m 1-\frac{2\mathrm{e}^{-2t}}{\mathrm{e}^{-2t}\left(\mathrm{e}^{2t}+1\right)} \, \mathrm{d}t\right)$ $= 5\left(\int_0^m 1-\frac{2\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}} \, \mathrm{d}t\right)$ $= 5\left[t+\ln(1+\mathrm{e}^{-2t})\right]_0^m$ $= 5\left[m+\ln(1+e^{-2m})-\ln(2)\right]$ $= 5\left[\ln(e^m)+\ln(1+e^{-2m})-\ln(2)\right]$ $= 5\left[\ln\left(\frac{e^m(1+e^{-2m})}{2}\right)\right]$ $= 5\ln\left(\frac{e^m+e^{-m}}{2}\right)$ (shown) **Alternative method:** $\int_0^m v \, \mathrm{d}t$ $= \int_0^m \frac{5(\mathrm{e}^{2t}-1)}{(\mathrm{e}^{2t}+1)} \, \mathrm{d}t$ $= 5\left(\int_0^m \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} \, \mathrm{d}t - \int_0^m \frac{1}{\mathrm{e}^{2t}+1} \, \mathrm{d}t\right)$ $= 5\left(\int_0^m \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} \, \mathrm{d}t - \int_0^m \frac{\mathrm{e}^{-2t}}{\mathrm{e}^{-2t}\left(\mathrm{e}^{2t}+1\right)} \, \mathrm{d}t\right)$ $= 5\left(\frac{1}{2}\int_0^m \frac{2\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} \, \mathrm{d}t - \int_0^m \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}} \, \mathrm{d}t\right)$ $= 5\left[\frac{1}{2}\ln\left(\mathrm{e}^{2t}+1\right)\right]_0^m + \frac{5}{2}\int_0^m \frac{-2\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}} \, \mathrm{d}t$ $= 5\left[\frac{1}{2}\ln\left(\mathrm{e}^{2t}+1\right)+\frac{1}{2}\ln(1+\mathrm{e}^{-2t})\right]_0^m$

	$= \frac{5}{2}\left[\ln\left(e^{2t}+2+e^{-2t}\right)\right]_0^m$ $= \frac{5}{2}\ln\left(\frac{e^{2m}+2+e^{-2m}}{4}\right)$ $= \frac{5}{2}\ln\left(\frac{\left(e^{m}+e^{-m}\right)^2}{2^2}\right)$ $= 5\ln\left(\frac{e^{m}+e^{-m}}{2}\right)$ (shown) $\because \frac{e^{m}+e^{-m}}{2} > 0$

	Suggested Solution
11(a)	The liquid level could either be in the cylindrical or conical section. Assume it's in the cylindrical section of height L and constant cross-sectional area of 16π $V = \pi r^2 L = 16\pi L$ $\frac{dV}{dt} = (16\pi)\frac{dL}{dt}$ $\Rightarrow \frac{dL}{dt} = \frac{-50}{16\pi}$ $= -0.99472$ cm/h for $h \geq 4$ Since $\frac{dL}{dt} = -0.99472 \neq -1.21$ $\Rightarrow$ liquid level is in the conical section $\Rightarrow h < 4$  Let r be radius of water surface in conical section. From diagram, $r = h$. $V = \frac{1}{3}\pi(r)^2 h = \frac{1}{3}\pi h^3$ $\frac{dV}{dh} = \pi h^2$ $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$ $-50 = \pi h^2(-1.21)$ $h = 3.6267$ $h = 3.63$ cm (3 sf)

	$V=\frac{1}{3}\pi\left(3.6267\right)^3$ $=49.9552$ $=50.0\text{ cm}^3$ (3 sf)
(b)	$\frac{k}{BD}=\tan\theta \Rightarrow BD=k\cot\theta$ $AB=m-k\cot\theta$ $R_{AB}=\mu\left(\frac{m-k\cot\theta}{r_1^4}\right)$ $\frac{k}{BC}=\sin\theta \Rightarrow BC=k\text{cosec}\theta$ $R_{BC}=\mu\left(\frac{k\text{cosec}\theta}{r_2^4}\right)$ Therefore, $R_T=\mu\left(\frac{m-k\cot\theta}{r_1^4}+\frac{k\text{cosec}\theta}{r_2^4}\right)$ (shown)
(c)	$\frac{\text{d}R_T}{\text{d}\theta}=\mu\left[-\frac{k\left(-\text{cosec}^2\theta\right)}{r_1^4}+\frac{k\left(-\text{cosec}\theta\cot\theta\right)}{r_2^4}\right]$ $=\mu k\left(\frac{\text{cosec}^2\theta}{r_1^4}-\frac{\text{cosec}\theta\cot\theta}{r_2^4}\right)$ $=\mu k\left(\frac{\text{cosec}^2\theta}{r_1^4}-\frac{\text{cosec}^2\theta\cos\theta}{r_2^4}\right)$ $=\mu k\text{cosec}^2\theta\left(\frac{1}{r_1^4}-\frac{\cos\theta}{r_2^4}\right)$ When $\frac{\text{d}R_T}{\text{d}\theta}=0$, $\mu k\text{cosec}^2\theta\left(\frac{1}{r_1^4}-\frac{\cos\theta}{r_2^4}\right)=0$ $\text{cosec}\theta=0$ (no soln.) or $\frac{1}{r_1^4}-\frac{\cos\theta}{r_2^4}=0$ $\cos\theta=\frac{r_2^4}{r_1^4}$ (shown)

(d)	<u>**Method 1**</u> $\frac{\mathrm{d}R_T}{\mathrm{d}\theta} = \mu k \operatorname{cosec}^2\theta\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right)$ $= \frac{\mu k \operatorname{cosec}^2\theta}{r_1^4 r_2^4}\left(r_2^4 - r_1^4\cos\theta\right)$ $= \frac{\mu k \operatorname{cosec}^2\theta}{r_2^4}\left(\frac{r_2^4}{r_1^4} - \cos\theta\right)$ As $\cos\theta^- > \cos\theta > \cos\theta^+$ when θ is acute,

θ^-	θ	θ^+
$\cos\theta^- > \frac{r_2^4}{r_1^4}$	$\cos\theta = \frac{r_2^4}{r_1^4}$	$\cos\theta^+ < \frac{r_2^4}{r_1^4}$
$\frac{\mathrm{d}R_T}{\mathrm{d}\theta} < 0$	$\frac{\mathrm{d}R_T}{\mathrm{d}\theta} = 0$	$\frac{\mathrm{d}R_T}{\mathrm{d}\theta} > 0$

$\therefore$ Stationary value in part (c) is a minimum i.e. least resistance.

<u>**Method 2**</u>

$$\frac{\mathrm{d}^2R_T}{\mathrm{d}\theta^2} = \mu\left[-\frac{k\left(2\operatorname{cosec}\theta\right)\left(-\operatorname{cosec}\theta\cot\theta\right)}{r_1^4} - \frac{k\left(-\operatorname{cosec}\theta\cot^2\theta + \operatorname{cosec}\theta\left(-\operatorname{cosec}^2\theta\right)\right)}{r_2^4}\right]$$

$$= \mu k\left[-\frac{2\operatorname{cosec}^2\theta\cot\theta}{r_1^4} + \frac{\operatorname{cosec}\theta\cot^2\theta + \operatorname{cosec}^3\theta}{r_2^4}\right]$$

$$= \mu k\left[-\frac{2}{r_1^4}\left(\frac{1}{\sin^2\theta}\cdot\frac{\cos\theta}{\sin\theta}\right) + \frac{1}{r_2^4}\left(\frac{1}{\sin\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\sin^3\theta}\right)\right]$$

$$= \frac{\mu k}{\sin^3\theta}\left[\frac{1}{r_2^4}\left(1+\cos^2\theta\right) - \frac{2}{r_1^4}\cos\theta\right]$$

When θ is acute and $\cos\theta = \frac{r_2^4}{r_1^4}$, $\frac{\mu k}{\sin^3\theta} > 0$ and

	$\frac{1}{r_2^4}\left(1+\cos^2\theta\right)-\frac{2}{r_1^4}\cos\theta=\frac{1}{r_2^4}\left(1+\frac{r_2^8}{r_1^8}\right)-\frac{2}{r_1^4}\cdot\frac{r_2^4}{r_1^4}$ $=\frac{1}{r_2^4}+\frac{r_2^4}{r_1^8}-2\frac{r_2^4}{r_1^8}$ $=\frac{1}{r_2^4}-\frac{r_2^4}{r_1^8}$ $=\frac{r_1^8-r_2^8}{r_1^8 r_2^4}$ >0 (since $r_1>r_2 \Rightarrow r_1^8>r_2^8$) Hence, $\frac{\mathrm{d}^2 R_T}{\mathrm{d}\theta^2}>0 \Rightarrow$ Minimum value.