HWA CHONG INSTITUTION

2022 JC2 Preliminary Examination

Higher 2

9758/02

MATHEMATICS

16 September 2022

3 hours

Candidate Name	

CT Group	21

Centre Number	S				

Index Number				

Write here how many additional pieces of writing paper you have used (if any).	

Candidates answer on the Question Paper.

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Do not write anything on the List of Formulae (MF26).

Write in dark blue or black pen. You may use HB pencil for any diagrams or graphs. Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Write your answers in the spaces provided in the Question Paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part of question.

For Examiner's Use			
Qn	**Marks**	**Total**	**Remarks**
1		4	
2		6	
3		7	
4		9	
5		14	
6		6	
7		8	
8		9	
9		11	
10		13	
11		13	
		100	

Remarks
- **a)** **INSTR**: Follow **instructions** as stated in Question (e.g. correct s.f , exact values, coordinates, similar form etc.)
- **b)** **NOT**: Correct Mathematical **Notations**
- **c)** **ACC**: **Accuracy** of Answers (e.g. affected by early rounding off, not writing +C for indefinite integrals etc.)

Section A: Pure Mathematics [40 marks]

1 State a sequence of 3 transformations that will transform the curve with equation $y = x^2$ onto the curve with equation $y = -x^2 + 3x - 4$. [4]

2 A curve C has parametric equations

$$x = \sin t \tan t, \qquad y = \cos t, \qquad \text{where } 0 \le t < \frac{\pi}{2}.$$

(i) Sketch C, stating the equation(s) of any asymptote(s) and coordinates of any axial intercept(s). [2]

(ii) The region A is bounded by C, the y-axis and the lines $y = \frac{1}{2}$ and $y = \frac{1}{\sqrt{2}}$.

Find the exact area of A. [4]

3 The complex number z is given by $z = 2(\cos\beta + \mathrm{i}\sin\beta)$ where $0 < \beta < \frac{\pi}{2}$.

(i) Show that $\frac{z}{4 - z^2} = (k \operatorname{cosec} \beta)\mathrm{i}$, where k is a positive real constant to be determined. [3]

(ii) Given that the complex number $w = -\sqrt{3} + \mathrm{i}$, find the three smallest positive integer values of n such that $\left(\frac{z}{4 - z^2}\right)(w^*)^n$ is a real number. [4]

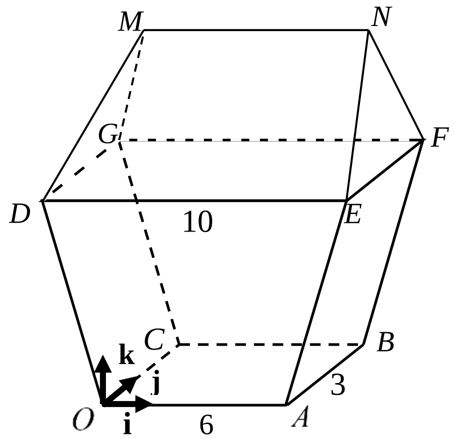
4 **(a)** **(i)** Find $\sum_{r=1}^{k}\left[\left(-\frac{1}{2}\right)^{r+1} + \ln(r+1)\right]$ in terms of k. Simplify your answer.

[4]

(ii) Hence determine if $\sum_{r=1}^{\infty}\left[\left(-\frac{1}{2}\right)^{r+1} + \ln(r+1)\right]$ exists. [1]

(b) The first term of an arithmetic series is positive. The sum of the first 6 terms of the series is 4.5, and the product of the first four terms of the series is 0. Find the 13th term of the series. [4]

5 The diagram below shows a 3-dimensional structure in which a pentahedron $DEFGMN$ lies on top of a trapezoidal prism $OABCDEFG$. Taking O as the origin, perpendicular vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel to OA and OC respectively. The base of the structure sits on the horizontal x–y plane.



Planes $OABC$ and $DEFG$ are parallel to each other. It is given that OC, AB, DG and EF are parallel to one another where $OC = AB = DG = EF = 3$ units. It is also given that OA, CB, DE, GF and MN are parallel to one another where $OA = CB = MN = 6$ units and $DE = GF = 10$ units. The pentahedron $DEFGMN$ and the trapezoidal prism $OABCDEFG$ each has a height of 12 units.

(i) The point D has coordinates $(-2, 0, s)$. State the value of s. [1]

(ii) The line DM is parallel to the vector $\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$ and the plane $ABFE$ has equation $6x = 36 + z$. It is given that the line DM does not intersect with the plane $ABFE$. Find the value of t. [2]

(iii) Show that the equation of the plane DGM is given by $\mathbf{r} \bullet \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix} = k$, where k is a constant to be determined. [2]

(iv) Find the acute angle between the planes DGM and $DEFG$. [3]

(v) Find the coordinates of M and the exact shortest distance from M to the plane $ABFE$. [4]

(vi) Another plane $\prod$ has cartesian equation $x = c$, where c is a constant. If the three planes $OAED$, EFN and $\prod$ all intersect at the point E, find the value of c, showing your working clearly. [2]

Section B: Probability and Statistics [60 marks]

6 Toddler Roy is playing with a shape sorter toy set which has a box that has a triangle hole and a square hole. The toy set also comes with a triangle block and a square block. The box will light up if the triangle block is correctly placed into the triangle hole.

(a) Roy picks one of the two blocks and place it into the shape sorter box. There is a probability of 0.9 that Roy will correctly place the chosen block into the hole of the same shape. If he correctly places a chosen block into the shape sorter box, there is a probability of 0.4 that it is a square block. Find the probability that the shape sorter box will not light up. [3]

(b) Roy's mother took the two blocks that Roy has and adds 4 more identical triangle blocks and *r* more identical square blocks, where $r \geq 3$. All the blocks are given to his sister, Joy, to randomly select 5 blocks without replacement. Suppose the probability of Joy choosing exactly 2 square blocks is twice the probability of Joy choosing exactly 4 square blocks, find the value of *r*. [3]

7 A company uses a machine to produce chocolate bars. The machine is designed to produce chocolate bars with average fat content of 30 g.

After using the machine for many years, the manager wishes to test, at 5% level of significance, if the machine still maintains the average fat content in the chocolate bars at 30 g. He selects a sample of 40 chocolate bars for testing.

(i) State, giving a reason, whether it is necessary to assume that the fat content of the chocolate bars is normally distributed for the test to be valid. [1]

After some years of usage, the machine broke down. While waiting for the new machine to arrive, the company reverted to their traditional mode of producing handmade chocolate bars. The manager suspects that handmade chocolate bars have higher average fat content than those made by the machines. To verify his suspicion, the manager asked his staff to perform a hypothesis test at 5% significance level, on a random sample of 40 handmade chocolate bars.

The fat content, *x* grams, of the 40 handmade chocolate bars are summarised as follows:

$$\sum x = 1220, \qquad \sum (x - 30.5)^2 = 50.$$

(ii) Calculate the unbiased estimates of the population mean and variance of the fat content in handmade chocolate bars. [2]

(iii) The sample mean fat content of the handmade chocolate bars is denoted by $\overline{x}$ grams. State the null and alternate hypotheses and calculate the range of values of $\overline{x}$ for which the null hypothesis would be rejected at 5% level of significance. Hence conclude if the manager's suspicion is valid at 5% significance level. [4]

(iv) The manager now knows the population variance of the fat content in handmade chocolate bars. It is given that the population variance is smaller than the unbiased estimate of the population variance calculated in part **(ii)**. Without carrying out another hypothesis test, explain with justification, if there will be a change in the manager's conclusion about his suspicion at 5% significance level. [1]

8 A school canteen committee consists of 4 parents, 2 student leaders and 4 teachers, chosen from 10 parents, 5 student leaders and 8 teachers.

(a) There is a married couple amongst the 10 parents. How many different canteen committees can be formed if the couple cannot serve on the committee together? [3]

The school canteen committee of 10 members has been formed.

(b) All members are to stand in a row to take a group photo with the Vice-Principal. Find the number of arrangements such that the Vice-Principal stands at the centre, both ends of the row are occupied by the students' leaders, and no two parents stand next to each other. [3]

(c) The committee members, together with the Vice-Principal, are seated at a round table with 11 chairs during lunch time. Find the probability that the parents are seated together and the teachers are separated. [3]

9 Abel has 1 white bag of marbles and 1 black bag of cards and he uses them to create a game. The white bag contains 2 red marbles, x blue marbles and $2x-1$ yellow marbles, where $x>1$. The black bag has 3 cards, and the cards are numbered with the number '0', '1' and '2' respectively.

In each round, Abel will first choose a marble randomly from the white bag. A red marble will give a score of 4. If a non-red marble is chosen, Abel will then choose a card randomly from his black bag. The score will then be twice of the number shown on the card drawn.

(i) Find the probability that Abel will get a score of 4 in a round of the game. Leave your answer in terms of x. [2]

(ii) Find the probability distribution of Abel's score in a round of the game in terms of x. [2]

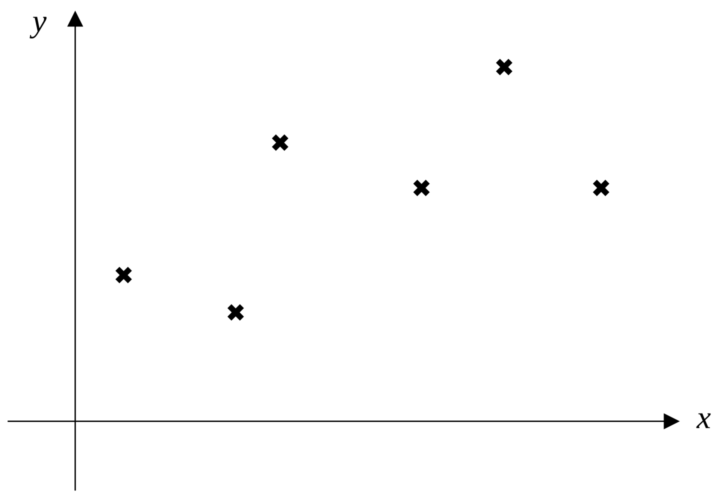
(iii) State the mode of Abel's score in a round of the game. [1]

(iv) Show that the variance of Abel's score in a round of the game is

$$\frac{60x+76}{3(3x+1)}-\frac{36(x+1)^2}{(3x+1)^2}.$$ [3]

(v) Given that $x=3$ and that Abel plays 100 rounds of the game, find the probability that the average score is not less than 2.5. [3]

10 **(a)** Draw the regression line of y on x and x on y on the diagram below, indicating the residual of one of the data points for each line. Hence describe the difference between the regression line of y on x and x on y. [3]



(b) Table A below shows the duration, t minutes, a diver can stay at different depths, d feet, below sea level.

Table A

d	50	60	70	80	90	100
t	80	55	45	35	25	22

(i) Sketch a scatter diagram of the data. [1]

(ii) Using the scatter diagram in part **(i)**, explain which of the following three models below is the most appropriate model for modelling the relationship between d and t.

(I) $t = ad + b$ where $a < 0$,

(II) $t = a\left(\frac{1}{d}\right) + b$ where $a > 0$, or

(III) $t = ae^{d} + b$ where $a < 0$.

State the equation of the regression line and the product moment correlation coefficient for the model, leaving your answers correct to 3 decimal places. [4]

(iii) Use your equation of the regression line from part **(ii)** to estimate the duration the diver can stay when he is 150 feet below sea level. Explain whether your estimate is reliable. [2]

(iv) A distance of 1 metre is equivalent to 3.28 feet. Re-write your equation from part **(ii)** so that it can be used to estimate the duration the diver can stay when he is at depth, D metres, below sea level. [1]

(v) A new data pair (d', t') is added to the data set given in Table A. If the product moment correlation coefficient found in part **(ii)** does not change with the addition of (d', t'), find a possible (d', t'). [2]

11 A cafeteria installed a vending machine which dispenses two types of coffee into disposable cups as follows:

(I) Black coffee, X ml, normally distributed with mean μ_1 ml and standard deviation 11.83 ml, or

(II) White coffee, by first releasing a quantity of black coffee, Y ml, normally distributed with mean μ_2 ml and standard deviation 11.83 ml and then adding milk, M ml, normally distributed with mean 35 ml and standard deviation 5.92 ml.

Given that $\mathrm{P}(X < 175) = \mathrm{P}(Y > 150)$, show that $\mu_1 + \mu_2 = 325$. [2]

For the rest of this question, assume that $\mu_1 = 180$.

(i) Find the probability that the total volume of 2 randomly chosen cups of black coffee exceeds twice the volume of a randomly chosen cup of white coffee by less than 15 ml. State an assumption that is needed for your calculation to be valid. [4]

(ii) The black coffee is sold at $4 per cup while the white coffee is sold at $5 per cup. The cost of ingredients for the black coffee is 1 cent per ml and the cost of ingredients for the milk is 2 cents per ml. 100 cups of coffee are sold per day, n of which are black coffee and the rest are white coffee. Find the largest value of n such that the probability of the total profit earned per day exceeding $230 is at least 0.8. [4]

To boost the sales of coffee from the vending machine, if the volume of the coffee dispensed falls below a certain level, the customer receives the drink free of charge. It is given that p% of the customers who selected black coffee received the drink free of charge.

From a large number of customers who selected black coffee, three customers are chosen at random.

(iii) State, in terms of p, the probability that exactly one of the three customers receives the drink free of charge. [1]

(iv) Given that the probability of exactly one of the three customers receiving the drink free of charge is at most 0.1, find the range of values of p. [2]