H

millennia institute

CANDIDATE NAME

CLASS

ADMISSION NUMBER

# 2019 Preliminary Exams
## Pre-University 3

**MATHEMATICS** **9758/02**

Paper 2 **18 September 2019**

**3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

**READ THESE INSTRUCTIONS FIRST**

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

| Qn No. | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 | Q9 | Q10 | Q11 | * | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Score | | | | | | | | | | | | | |
| Max Score | 3 | 7 | 9 | 9 | 12 | 5 | 8 | 11 | 11 | 12 | 13 | | 100 |

This document consists of **25** printed pages.

**[Turn over**

## Section A: Pure Mathematics [40 marks]

**1** It is known that the $n$th term of a sequence is given by

$$p_n = 3^{n-1} + a,$$

where $a$ is a constant.

Find $\sum_{r=3}^{n} p_r$. [3]

**2** **(i)** An architect places 25 rectangular wooden planks in a row as a design for part of the facade of a building. The lengths of the first 18 planks form an arithmetic progression and the first plank has length 4 m. Given that the sum of the first three planks is 11.46 m, find the length of the 18th plank. [2]

**(ii)** For the last 7 wooden planks, each plank has a length that is $\frac{5}{4}$ of the length of the previous plank. Find the length of the 25th plank. [2]

**(iii)** The even-numbered planks (2nd plank, 4th plank, 6th plank and so on) are painted blue. Find the total length of the planks that are painted blue. [3]

**3** **(a)** Find the general solution for the following differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{e}^{-5x+3} + \sin x.$$ [3]

**(b)** By using the substitution $z = x + \frac{\mathrm{d}y}{\mathrm{d}x}$, show that the following differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - x + 1 = 0$$

can be reduced to

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z.$$ [2]

Hence, given that when $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, find $y$ in terms of $x$. [4]

**4** The curve $C$ has parametric equations

$$x = t^3 - t^2, \quad y = t^2 + 2t - 3,$$

where $t \geq -1$.

**(i)** Sketch the curve $C$, stating clearly any axial intercepts and the coordinates of any end-points. [2]

The point $P$ on the curve $C$ has parameter $t = 2$.

**(ii)** Without using a calculator, find the equation of tangent to $C$ at $P$. [3]

**(iii)** Without using a calculator, find the area of the region bounded by $C$, the tangent to $C$ at $P$ and the $y$-axis. [4]

**5** **(a)** One of the roots of the equation $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ is $b$i, where $a$ and $b$ are positive real constants.

**(i)** Find the values of $a$ and $b$ and hence find the other roots of the equation. [4]

**(ii)** Deduce the roots of the equation $w^4 + 2w^3 + aw^2 + 8w + 40 = 0$. [2]

**(b)** **(i)** The complex number $-6 + \left(2\sqrt{3}\right)\mathrm{i}$ is denoted by $w$. Without using a calculator, find an exact expression of $w^n$ in modulus-argument form. [3]

**(ii)** Hence find the two smallest positive integers $n$ such that $w^n w^*$ is purely imaginary. [3]

**Section B: Probability and Statistics [60 marks]**

**6** The government of Country $X$ uses a particular method to create a unique Identification Number (IN) for each of its citizens. The IN consists of 4 digits from 0 to 9 followed by one of the ten letters A – J. The digits in the IN can be repeated.

**(i)** Find the number of different Identification Numbers that can be created. [1]

Suppose the letter at the end of the IN is determined by the following steps:

Step 1: Find the sum of all the digits in the IN.

Step 2: Divide the sum by 10 and note the remainder.

Step 3: Use the table below to determine the letter that corresponds to the remainder.

| **Remainder** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Letter** | A | B | C | D | E | F | G | H | I | J |

**(ii)** It is known that all citizens who are born in the year 1990 must have the digit '9' appearing twice in their IN and the sum of all the digits in their IN is at least 20. Eric, who is born in the year 1990, has 'I' as the last letter of his IN. Find the number of INs that could possibly belong to Eric. [4]

**7** In a carnival, a player begins a game by rolling a fair 12-sided die which consists of 3 red faces, 7 blue faces and 2 white faces. When the die thrown comes to rest, the colour of the uppermost face of the die is noted. If the colour of the uppermost face is red, a ball is picked from box $A$ that contains 3 red and 4 blue balls. If the colour of the uppermost face is blue, a ball is picked from box $B$ that contains 2 red, 3 blue and 3 green balls. Otherwise, the game ends. A mystery gift is only given when the colour of the uppermost face of the die is the same as the colour of the ball picked.

It is assumed that Timothy plays the game only once.

**(i)** Find the probability that the colour of the uppermost face of the die is blue and a red ball is picked. [1]

**(ii)** Find the probability the mystery gift is given to Timothy. [2]

**(iii)** Given that Timothy did not win the mystery gift, find the probability that a red ball is picked. [3]

One of the rules of the game is changed such that if the colour of the uppermost face is white when the die thrown comes to rest, the player gets to roll the die again. The other rules of the game still hold.

Alicia plays the game once. Find the probability that she obtains the mystery gift in the end. [2]

**8** **(a)** Sketch a scatter diagram that might be expected when $x$ and $y$ are related approximately as given in each of the cases (A), (B) and (C) below. In each case your diagram should include 6 points, approximately equally spaced with respect to $x$, with all $x$- and $y$-values positive. The letters $a$, $b$, $c$, $d$, $e$ and $f$ represent constants.

(A) $y = a + bx^2$, where $a$ and $b$ are positive.

(B) $y = c + d \ln x$, where $c$ is positive and $d$ is negative.

(C) $y = e + \dfrac{f}{x}$, where $e$ is positive and $f$ is negative. [3]

**(b)** The following table shows the population of a certain country, $P$, in millions, at various times, $t$ years after the year 1990.

| $t$ | 4 | 10 | 12 | 14 | 15 | 18 | 20 | 24 |
|---|---|---|---|---|---|---|---|---|
| $P$ | 3.65 | 3.89 | 3.95 | 4.02 | 4.15 | 4.30 | 4.45 | 4.95 |

**(i)** Draw the scatter diagram for these values, labelling the axes. Give a reason why a linear model may not be appropriate. [2]

**(ii)** Explain which of the three cases in part **(a)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]

**(iii)** A statistician wants to predict the population of the country in the year 2020. Use the case that you identified in part **(b)(ii)** to find the equation of a suitable regression line, and use your equation to find the required prediction. [3]

**(iv)** Comment on the reliability of the statistician's prediction. [1]

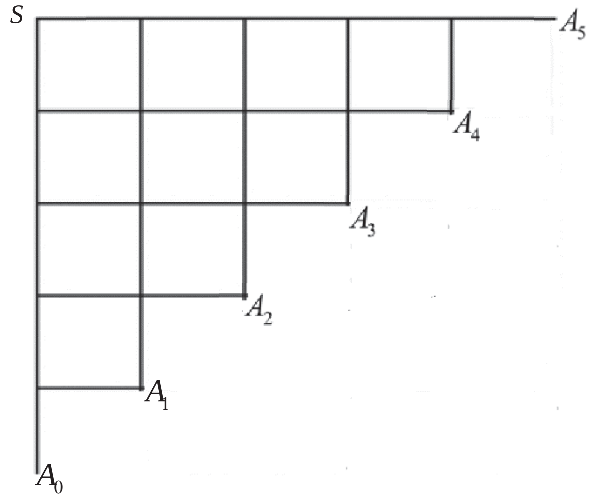
**9** A chemist is conducting experiments to analyse the amount of active ingredient $A$ in a particular type of health supplement tablets. Each tablet is said to contain an average amount of 50 mg of active ingredient $A$. A random sample of 40 tablets is taken and the amount of active ingredient $A$ per tablet is recorded. The sample mean is 50.6 mg and the sample variance is 2.15 mg$^2$ respectively.

**(i)** Explain the meaning of 'a random sample' in the context of the question. [1]

**(ii)** Test, at the 1% level of significance, whether the mean amount of active ingredient $A$ in a tablet has changed. You should state your hypotheses and define any symbols you use. [6]

In a revised formula of the same type of health supplement tablets, the amount of active ingredient $A$ now follows a normal distribution with population variance 1.5 mg$^2$. The chemist wishes to test his claim that the mean amount of active ingredient $A$ per tablet is more than 50 mg. A second random sample of $n$ such tablets is analysed and its mean is found to be 50.4 mg. Find the set of values that $n$ can take such that the chemist's claim is valid at 2.5% level of significance. [4]

**10**




A particle moves one step each time either to the right or downwards through a network of connected paths as shown above. The particle starts at $S$, and, at each junction, randomly moves one step to the right with probability $p$, or one step downwards with probability $q$, where $q = 1 - p$. The steps taken at each junction are independent. The particle finishes its journey at one of the 6 points labelled $A_i$, where $i = 0, 1, 2, 3, 4, 5$ (see diagram). Let $\{ X = i \}$ be the event that the particle arrives at point $A_i$.

**(i)** Show that $\mathrm{P}(X = 2) = 10p^2q^3$ . [2]

**(ii)** After experimenting, it is found that the particle will end up at point $A_2$ most of the time. By considering the mode of $X$ or otherwise, show that $\frac{1}{3} < p < \frac{1}{2}$. [4]

The above setup is a part of a two-stage computer game.

- If the particle lands on $A_0$, the game ends immediately and the player will not win any points.
- If the particle lands on $A_i$, where $i$ = 2 or 4, then the player gains 2 points.
- If the particle lands on $A_i$, where $i$ = 1, 3 or 5, then the player proceeds on to the next stage, where there is a probability of 0.4 of winning the stage. If he wins the stage, he gains 5 points; otherwise he gains 3 points.

Let $Y$ be the number of points gained by the player when one game is played.

**(iii)** If $p = 0.4$, determine the probability distribution of $Y$. [4]

**(iv)** Hence find the expectation and variance of $Y$. [2]

**11** The two most popular chocolates sold by the *Dolce* chocolatier are the dark truffles and salted caramel ganaches and their masses have independent normal distributions. The masses, in grams, of dark truffles have the distribution $\mathrm{N}(17, 1.3^2)$.

**(i)** Find the probability that the total mass of 4 randomly chosen dark truffles is more than 70 g. [2]

**(ii)** The dark truffles are randomly packed into boxes of 4. In a batch of 20 boxes, find the probability that there are more than 3 boxes of dark truffles that have a mass more than 70 g. State an assumption you made in your calculations. [4]

The masses of salted caramel ganaches are normally distributed such that the proportion of them having a mass less than 12 grams is the same as the proportion of them having a mass greater than 15 grams. It is also given that 97% of the salted caramel ganaches weigh at most 15 grams.

**(iii)** Find the mean and variance of the masses of salted caramel ganaches. [3]

Dark truffles are sold at \$0.34 per gram and the salted caramel ganaches are sold at \$0.28 per gram.

**(iv)** Find the probability that the total cost of 6 randomly chosen salted caramel ganaches is less than the total cost of 4 randomly chosen dark truffles. State the distribution you use and its parameters. [4]

**End of Paper**