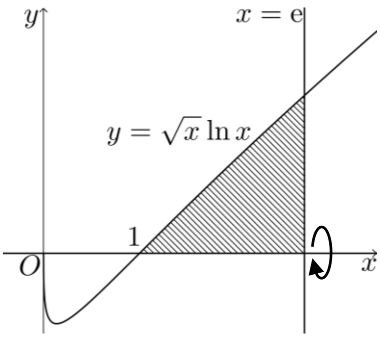
Question 1 (Integration by Parts, Volume of Revolution)

(i) $I_n = \int_1^{e} x(\ln x)^n \, dx$ $= \left[\frac{x^2}{2}(\ln x)^n\right]_1^{e} - \int_1^{e} \frac{x^2}{2} \cdot n(\ln x)^{n-1} \frac{1}{x} dx$ $= \frac{e^2}{2} - \frac{n}{2}\int_1^{e} x(\ln x)^{n-1} \, dx$ $= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ (shown)	
(ii) Volume $= \pi\int_1^{e} x(\ln x)^2 \, dx$ $= \pi I_2$ $= \pi\left(\frac{e^2}{2} - \frac{2}{2}I_1\right)$ $= \pi\left[\frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{2}I_0\right)\right]$ $= \frac{\pi}{2}I_0 = \frac{\pi}{2}\left[\frac{x^2}{2}\right]_1^{e}$ $= \frac{\pi}{4}(e^2 - 1)$ units^3 	

Question 2 (Inequalities)

(i)

$$\frac{px^2-1}{x^2+(1-p)x-p}\geq 1$$

$$\Rightarrow \frac{px^2-1}{x^2+(1-p)x-p}-1\geq 0$$

$$\Rightarrow \frac{px^2-1-\left[x^2+(1-p)x-p\right]}{x^2+(1-p)x-p}\geq 0$$

$$\Rightarrow \frac{px^2-1-x^2+(p-1)x+p}{x^2-px+x-p}\ p\geq 0$$

$$\Rightarrow \frac{(p-1)x^2+(p-1)x+(p-1)}{(x+1)(x-p)}\geq 0$$

$$\Rightarrow \frac{(p-1)\left(x^2+x+1\right)}{(x+1)(x-p)}\geq 0$$

$$\Rightarrow \frac{x^2+x+1}{(x+1)(x-p)}\geq 0 \quad \text{since } p>1\Rightarrow p-1>0$$

$$\Rightarrow \frac{1}{(x+1)(x-p)}\geq 0 \quad \text{since } x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0$$

$$\Rightarrow (x+1)(x-p)>0 \quad \text{since denominator cannot be zero}$$

$$\Rightarrow x<-1 \quad \text{or} \quad x>p$$

(ii) We observe that

$$\frac{px^2-1}{x^2+(p-1)|x|-p}=\frac{p(-|x|)^2-1}{(-|x|)^2+(1-p)(-|x|)-p}.$$

Hence we replace x by $-|x|$ in the answer obtained in **(i)**, i.e.

$$-|x|<-1 \quad \text{or} \quad -|x|>p \quad \Rightarrow \quad |x|>1 \quad \text{or} \quad |x|<-p \text{ (N.A.)}$$

$$\Rightarrow \quad x>1 \quad \text{or} \quad x<-1$$

Question 3 (Vectors I – Abstract Vectors)

(i)	Given that OAB is an equilateral triangle, $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert$ and $\angle AOB = \frac{\pi}{3}$. $\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB$ $= \lvert\mathbf{a}\rvert^2 \cos\left(\frac{\pi}{3}\right)$ $= \frac{1}{2}\lvert\mathbf{a}\rvert^2$	
(ii)	$\overrightarrow{AC}$ $= \overrightarrow{OC} - \overrightarrow{OA}$ $= 3\mathbf{a} - 4\mathbf{b} - \mathbf{a}$ $= 2\mathbf{a} - 4\mathbf{b}$ $\overrightarrow{OA}\cdot\overrightarrow{AC}$ $= \mathbf{a}\cdot(2\mathbf{a} - 4\mathbf{b})$ $= 2\mathbf{a}\cdot\mathbf{a} - 4\mathbf{a}\cdot\mathbf{b}$ $= 2\lvert\mathbf{a}\rvert^2 - 4\left(\frac{1}{2}\lvert\mathbf{a}\rvert^2\right)$ $= 2\lvert\mathbf{a}\rvert^2 - 2\lvert\mathbf{a}\rvert^2$ $= 0$ Since $\overrightarrow{OA}\cdot\overrightarrow{AC} = 0$, $\therefore$ OA and AC are perpendicular.	
(iii)	$\mathbf{r}\times(\mathbf{b}-\mathbf{a}) = \mathbf{a}\times(\mathbf{b}-\mathbf{a})$ $\mathbf{r}\times(\mathbf{b}-\mathbf{a}) - \mathbf{a}\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$ $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$ $\mathbf{r}-\mathbf{a} = \lambda(\mathbf{b}-\mathbf{a})$ $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) = \overrightarrow{OA} + \lambda\,\overrightarrow{AB}$ The equation represents the <u>line AB</u> and $\mathbf{r}$ represents the position vector of a variable point on this line.	

Question 4 (Complex Numbers – Algebra)

(i) <u>**Method 1 (recommended in this case as use of GC is allowed)**</u> Let $w=a+\mathrm{i}b$ and $u=c+\mathrm{i}d$, where a, b, c and d are real. $\therefore w^*=a-\mathrm{i}b$ and $u^*=c-\mathrm{i}d$ Since $w^*-2\mathrm{i}u=8$, $a-\mathrm{i}b-2\mathrm{i}(c+\mathrm{i}d)=8$ $a+2d-\mathrm{i}(b+2c)=8$ Comparing real and imaginary parts, $a+2d=8$ ---------- (1) $b+2c=0$ ---------- (2) Since $(2\mathrm{i}-1)w+2u^*=4$, $(2\mathrm{i}-1)(a+\mathrm{i}b)+2(c-\mathrm{i}d)=4$ $2\mathrm{i}a-2b-a-\mathrm{i}b+2c-2\mathrm{i}d=4$ $-a-2b+2c+\mathrm{i}(2a-b-2d)=4$ Comparing real and imaginary parts, $-a-2b+2c=4$ ---------- (3) $2a-b-2d=0$ ---------- (4) Using GC to solve the 4 linear equations, $a=2$, $b=-2$, $c=1$, $d=3$. $\therefore w=2-2\mathrm{i}$ and $u=1+3\mathrm{i}$	
<u>**Method 2**</u> Since $w^*-2\mathrm{i}u=8$, $w^*=8+2\mathrm{i}u \Rightarrow w=8^*+(2\mathrm{i}u)^*=8-2\mathrm{i}u^*$ (1) Substituting (1) into $(2\mathrm{i}-1)w+2u^*=4$, $(2\mathrm{i}-1)(8-2\mathrm{i}u^*)+2u^*=4$ $16\mathrm{i}-8+4u^*+2\mathrm{i}u^*+2u^*=4$ $(6+2\mathrm{i})u^*=12-16\mathrm{i}$ $u^*=\dfrac{12-16\mathrm{i}}{6+2\mathrm{i}}=1-3\mathrm{i}$ $u=1+3\mathrm{i}$ Thus, from equation (1), $w=8-2\mathrm{i}(1-3\mathrm{i})=8-2\mathrm{i}-6=2-2\mathrm{i}$	

(ii)	**<u>Method 1 (Conjugate Root Theorem)</u>** Since the coefficients of the equation are all real, $\therefore w^*$ and u^* are also roots of the equation. $(z-w)(z-w^*)(z-u)(z-u^*)=0$ $\left[z^2-(w+w^*)z+ww^*\right]\left[z^2-(u+u^*)z+uu^*\right]=0$ $\left[z^2-(2\operatorname{Re}(w))z+\lvert w\rvert^2\right]\left[z^2-(2\operatorname{Re}(u))z+\lvert u\rvert^2\right]=0$ $\left[z^2-2(2)z+2^2+2^2\right]\left[z^2-2(1)z+1^2+3^2\right]=0$ $(z^2-4z+8)(z^2-2z+10)=0$ $\therefore c=8,\ d=2$	
	<u>Method 2</u> $(z^2-4z+c)(z^2-dz+10)=0$ $z^2-4z+c=0$ or $z^2-dz+10=0$ $z=\frac{4\pm\sqrt{4^2-4c}}{2}$ or $z=\frac{d\pm\sqrt{d^2-40}}{2}$ $z=2\pm\sqrt{4-c}$ or $z=\frac{d\pm\sqrt{d^2-40}}{2}$ Comparing, $2-\sqrt{4-c}=w=2-2\mathrm{i}$ $\Rightarrow 4-c=2^2\Rightarrow c=4+2^2=8$ Therefore, $\frac{d}{2}=\operatorname{Re}(u)=1\Rightarrow d=2$	

Question 5 (Maxima & Minima Problems)

(i) $\frac{4}{3}\pi r^3+\pi r^2 x=600$ $\pi r^2\left(\frac{4}{3}r+x\right)=600$ $\frac{4}{3}r+x=\frac{600}{\pi r^2}$ $x=\frac{600}{\pi r^2}-\frac{4}{3}r$ $V=\frac{4}{3}\pi\left(r+\frac{1}{4}\right)^3+\pi\left(r+\frac{1}{4}\right)^2 x-600$ $=\frac{4}{3}\pi\left(r+\frac{1}{4}\right)^3+\pi\left(r+\frac{1}{4}\right)^2\left(\frac{600}{\pi r^2}-\frac{4}{3}r\right)-600$ $=\left(r+\frac{1}{4}\right)^2\left[\frac{4\pi}{3}\left(r+\frac{1}{4}\right)+\frac{600}{r^2}-\frac{4\pi}{3}r\right]-600$ $=\left(r+\frac{1}{4}\right)^2\left(\frac{\pi}{3}+\frac{600}{r^2}\right)-600$	
(ii) For V to be minimum, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$. $\frac{\mathrm{d}V}{\mathrm{d}r}=2\left(r+\frac{1}{4}\right)\left(\frac{\pi}{3}+\frac{600}{r^2}\right)+\left(r+\frac{1}{4}\right)^2\left[\frac{600(-2)}{r^3}\right]$ $=\left(2r+\frac{1}{2}\right)\left(\frac{\pi}{3}+\frac{600}{r^2}\right)-\frac{1200}{r^3}\left(r^2+\frac{1}{2}r+\frac{1}{16}\right)$ $=\frac{2\pi r}{3}+\frac{1200}{r}+\frac{\pi}{6}+\frac{300}{r^2}-\frac{1200}{r}-\frac{600}{r^2}-\frac{75}{r^3}$ $=\frac{2\pi r}{3}+\frac{\pi}{6}-\frac{300}{r^2}-\frac{75}{r^3}$ $\therefore \frac{2\pi r}{3}+\frac{\pi}{6}-\frac{300}{r^2}-\frac{75}{r^3}=0$ Using GC, $r=5.2322$ $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{2\pi}{3}-\frac{300(-2)}{r^3}-\frac{75(-3)}{r^4}$ $=\frac{2\pi}{3}+\frac{600}{r^3}+\frac{225}{r^4}$ Since $r>0$, $\frac{600}{r^3}>0$ and $\frac{225}{r^4}>0$, so $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}>0$ for all real $r>0$. $\therefore$ V is minimum when $r=5.23$ (3sf)	

Question 6 (Sequences and Series)

(i) $u_k - u_{k+1} = \frac{1}{k!} - \frac{1}{(k+1)!} = \frac{k+1-1}{(k+1)!} = \frac{k}{(k+1)!}$ $\frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \ldots + \frac{3n+2}{(3n+3)!}$ $= \sum_{r=3}^{3n+2} \frac{r}{(r+1)!}$ $= \sum_{r=3}^{3n+2} \left[\frac{1}{r!} - \frac{1}{(r+1)!} \right]$ $= \sum_{r=3}^{3n+2} \left[\frac{1}{r!} - \frac{1}{(r+1)!} \right]$ $= \frac{1}{3!} - \frac{1}{4!}$ $+ \frac{1}{4!} - \frac{1}{5!}$ $\vdots$ $+ \frac{1}{(3n+1)!} - \frac{1}{(3n+2)!}$ $+ \frac{1}{(3n+2)!} - \frac{1}{(3n+3)!}$ $= \frac{1}{3!} - \frac{1}{(3n+3)!}$ $= \frac{1}{6} - \frac{1}{(3n+3)!}$	

(ii) Replace r with $h+1$: $\sum_{r=5}^{3n+3}\frac{r-1}{r!}=\sum_{h=4}^{3n+2}\frac{h}{(h+1)!}$ $=\sum_{h=3}^{3n+2}\frac{h}{(h+1)!}-\frac{3}{4!}$ $=\frac{1}{6}-\frac{1}{(3n+3)!}-\frac{1}{8}$ $=\frac{1}{24}-\frac{1}{(3n+3)!}$ $\sum_{r=5}^{3n+3}\frac{3}{r!}<\sum_{r=5}^{3n+3}\frac{r-1}{r!}$ $=\frac{1}{24}-\frac{1}{(3n+3)!}$ $<\frac{1}{24}$	

Question 7 (Arithmetic and Geometric Series)

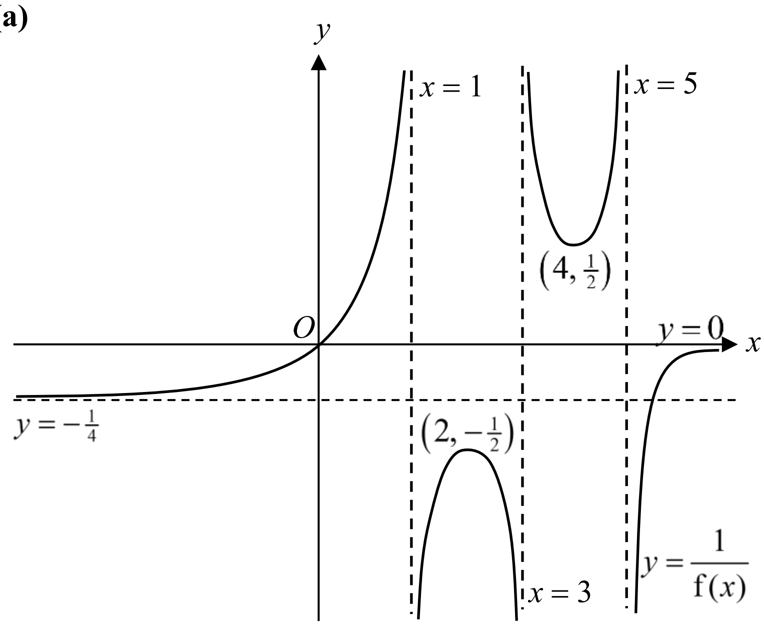
(a)	$160+(n-1)(-8)>0$ $168-8n>0$ $n<21$ Thus, Amy can cut off at most 20 pieces.	
(b)	$S_\infty \le 1000 \Rightarrow \dfrac{160}{1-p} \le 1000$ $\Rightarrow 160 \le 1000(1-p)$ $\Rightarrow 0 < p \le 0.84$ Largest value of p is 0.84. For the total length of ribbon that Bala cuts off to **not** exceed 9.5 m, $S_n \le 950 \quad \Rightarrow \dfrac{160\left(1-0.84^n\right)}{1-0.84} \le 950$ $\Rightarrow 160\left(1-0.84^n\right) \le 152$ $\Rightarrow 0.84^n \ge 0.05$ $\Rightarrow n \le \dfrac{\ln 0.05}{\ln 0.84}$ $\Rightarrow 0 < n \le 17.18$ So, Bala can cut off at most 17 pieces of string (since n is an integer).	
(c)	Total length of the original long roll of ribbon $=\dfrac{20}{2}\left[2(160)+19(-8)\right]+\dfrac{160\left(1-0.84^{17}\right)}{1-0.84}+4.5$ $=1680+948.388+4.5$ $=2632.888$ $\approx 2633\text{ cm }(\text{nearest cm})$	

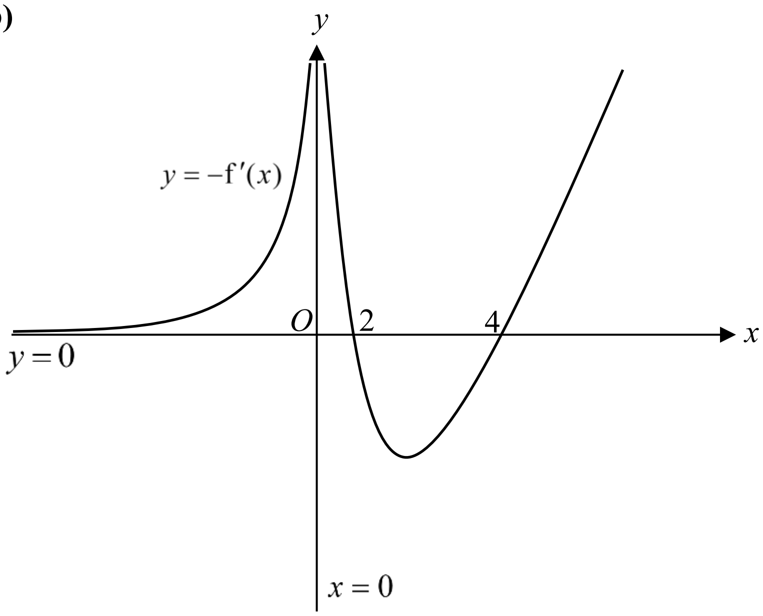
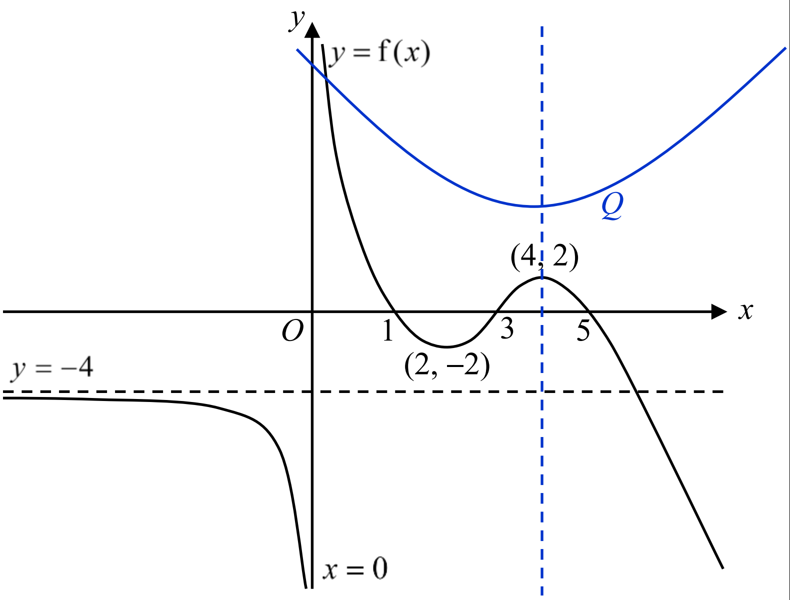
Question 8 (Complex Numbers – Geometrical Forms)

(i) $w^* = \left(\left(-\mathrm{i}\sqrt{3}\right)z\right)^* = \mathrm{i}\sqrt{3}z^*$ $= \sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)} 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}\right)} = 2\sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ $z + w^* = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)} + 2\sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ $= 2\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}\left(-\mathrm{e}^{\mathrm{i}\pi} + \sqrt{3}\right)$ $= 2\left(\sqrt{3} - 1\right)\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$	
(ii) Since $\left(z + w^*\right)^n$ is purely imaginary, then $\arg\left(\left(z + w^*\right)^n\right) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ $n\arg\left(z + w^*\right) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ $\frac{3n\pi}{4} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ $\frac{3n}{4} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}, \ldots$ $= \frac{1}{2}\left(2k+1\right)$, where $k \in \mathbb{Z}$ $n = \frac{2}{3}\left(2k+1\right)$, where $k \in \mathbb{Z}$	

(iii) $\arg\left(\dfrac{z+w^*}{z^*w}\right)=\arg\left(z+w^*\right)-\arg\left(z^*w\right)$ $=\arg\left(z+w^*\right)-\arg\left(z^*\right)-\arg\left(w\right)$ $=\dfrac{3\pi}{4}-\dfrac{\pi}{4}+\dfrac{3\pi}{4}$ $=\dfrac{5}{4}\pi\equiv-\dfrac{3}{4}\pi$ Thus, $\operatorname{Im}(v)=\operatorname{Re}(v)$ $\left\lvert\dfrac{z+w^*}{z^*w}\right\rvert=\dfrac{\lvert z+w^*\rvert}{\lvert z\rvert\lvert w\rvert}$ $=\dfrac{\lvert z+w^*\rvert}{\lvert z\rvert\left\lvert\left(-i\sqrt{3}\right)z\right\rvert}$ $=\dfrac{\lvert z+w^*\rvert}{\sqrt{3}\lvert z\rvert^2}$ $=\dfrac{2\left(\sqrt{3}-1\right)}{4\sqrt{3}}$ $=\dfrac{\sqrt{3}-1}{2\sqrt{3}}=\dfrac{1}{2}-\dfrac{\sqrt{3}}{6}$	

Question 9 (Transformations of Graphs)

(a)	
	

(b) 	
l.p. $y = p(x-4)^2 + q$  For the two curves to intersect exactly once, $q > 2$.	

Question 10 (Binomial Series, Geometric Series)

(i)	$\mathrm{f}(x)=(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n$ $\approx a^n\left[1+n\left(\frac{x}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{x}{a}\right)^4\right]$ $=a^n+na^{n-1}x+\frac{1}{2}n(n-1)a^{n-2}x^2+\frac{1}{6}n(n-1)(n-2)a^{n-3}x^3$ $+\frac{1}{24}n(n-1)(n-2)(n-3)a^{n-4}x^4$ Since the 1st, 3rd and 5th terms in the series expansion of $\mathrm{f}(x)$ are consecutive terms of a geometric series, $\frac{\frac{1}{2}n(n-1)a^{n-2}x^2}{a^n}=\frac{\frac{1}{24}n(n-1)(n-2)(n-3)a^{n-4}x^4}{\frac{1}{2}n(n-1)a^{n-2}x^2}$ $\frac{n(n-1)x^2}{2a^2}=\frac{(n-2)(n-3)x^2}{12a^2}$ $6n(n-1)=n^2-5n+6\ (\because a,\ x\neq 0)$ $5n^2-n-6=0$ $(5n-6)(n+1)=0$ $n=\frac{6}{5}$ or $n=-1$	
(ii)	For the Maclaurin series of $\mathrm{f}(x)$ to converge, $\left\lvert\frac{x}{a}\right\rvert<1\Rightarrow\lvert x\rvert<a.$ The common ratio of G is $\frac{-1(-1-1)x^2}{2a^2}=\frac{x^2}{a^2}.$ Thus, for G to be convergent, $\left\lvert\frac{x^2}{a^2}\right\rvert<1\Rightarrow\lvert x\rvert<a.$ Thus, the range of values of x for the Maclaurin series of $\mathrm{f}(x)$ to converge is equal to the range of values of x for G to converge.	
(iii)	Sum to infinity of $G=\frac{a^{-1}}{1-\frac{x^2}{a^2}}$ $=\frac{a}{a^2-x^2}$	
(iv)	$0<\lvert x\rvert<a\Rightarrow 0<x^2<a^2$ $\Rightarrow -a^2<-x^2<0$ $\Rightarrow 0<a^2-x^2<a^2$ $\Rightarrow \frac{1}{a^2-x^2}>\frac{1}{a^2}$ $\Rightarrow S=\frac{a}{a^2-x^2}>\frac{1}{a}$	

Question 11 (Vectors II – Lines and Planes)

(i) Since $\begin{pmatrix} 2/3 \\ c \\ 2/3 \end{pmatrix}$ is a unit vector, $\left|\begin{pmatrix} 2/3 \\ c \\ 2/3 \end{pmatrix}\right| = 1$. Therefore,

$$\sqrt{\frac{4}{9}+c^2+\frac{4}{9}} = 1 \Rightarrow \frac{8}{9}+c^2 = 1$$

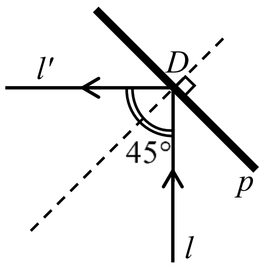
$$\Rightarrow c^2 = \frac{1}{9} \Rightarrow c = -\frac{1}{3}$$

$$\left(\text{reject } c = \frac{1}{3} \text{ since } c<0\right)$$

Thus, l has direction vector $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

$$\cos 45^\circ = \left|\frac{\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ -1 \end{pmatrix}}{\sqrt{2^2+(-1)^2+2^2}\sqrt{b^2+1}}\right|$$

$$\frac{1}{\sqrt{2}} = \left|\frac{-b-2}{3\sqrt{b^2+1}}\right|$$



$$\sqrt{\frac{b^2+1}{2}} = \left|\frac{b+2}{3}\right|$$

$$\frac{b^2+1}{2} = \left|\frac{b+2}{3}\right|^2 = \frac{b^2+4b+4}{9}$$

$$9b^2+9 = 2b^2+8b+8$$

$$7b^2-8b+1 = 0$$

$$(b-1)(7b-1) = 0$$

$$b = 1 \text{ or } b = \frac{1}{7}\left(\text{rej } \because\ b \in \mathbb{Z}\right)$$

(ii) Since D lies on l, $\overrightarrow{OD} = \begin{pmatrix} 5+2\lambda \\ 1-\lambda \\ 3+2\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

p has equation $y - z = 4 \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 4$

Since D lies on p, $\begin{pmatrix} 5+2\lambda \\ 1-\lambda \\ 3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 4.$

$$\begin{pmatrix} 5+2\lambda \\ 1-\lambda \\ 3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 1. \quad \Rightarrow 1-\lambda-3-2\lambda = 4$$

$$\Rightarrow 3\lambda = -6$$

$$\Rightarrow \lambda = -2$$

So $\overrightarrow{OD} = \begin{pmatrix} 5+2(-2) \\ 1-(-2) \\ 3+2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

Coordinates of D are $(1, 3, -1)$.

(iii) Let L be the normal to p passing through D and N be the foot of perpendicular of S onto L. L has equation $$\mathbf{r}=\begin{pmatrix}1\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}0\\1\\-1\end{pmatrix},\ \mu\in\mathbb{R}.$$ **<u>Method 1</u>** N lies on L: $\overrightarrow{ON}=\begin{pmatrix}1\\3+\mu\\-1-\mu\end{pmatrix}$ for some $\mu\in\mathbb{R}$. $\therefore\ \overrightarrow{SN}=\begin{pmatrix}1\\3+\mu\\-1-\mu\end{pmatrix}-\begin{pmatrix}5\\1\\3\end{pmatrix}=\begin{pmatrix}-4\\2+\mu\\-4-\mu\end{pmatrix}$ Since $SN\perp$ normal to p, $\overrightarrow{SN}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0$. Thus, $\begin{pmatrix}-4\\2+\mu\\-4-\mu\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0\Rightarrow 2+\mu+4+\mu$ $\Rightarrow 2\mu=-6$ $\Rightarrow \mu=-3$ So $\overrightarrow{ON}=\begin{pmatrix}1\\3+(-3)\\-1-(-3)\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$	

Method 2 $\overrightarrow{DN}$ is the projection vector of $\overrightarrow{DS}$ onto the normal to p. Thus, $\overrightarrow{DN}=\left(\dfrac{\overrightarrow{DS}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}}{1^2+(-1)^2+0^2}\right)\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $=\left(\dfrac{1}{2}\begin{pmatrix}4\\-2\\4\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)\begin{pmatrix}0\\1\\-1\end{pmatrix}=\dfrac{-2-4}{2}\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\-3\\3\end{pmatrix}$ $\therefore\overrightarrow{ON}=\begin{pmatrix}1\\3\\-1\end{pmatrix}+\begin{pmatrix}0\\-3\\3\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ By Ratio Theorem, $\overrightarrow{ON}=\dfrac{\overrightarrow{OS}+\overrightarrow{OS'}}{2}$. Thus, $\overrightarrow{OS'}=2\overrightarrow{ON}-\overrightarrow{OS}=2\begin{pmatrix}1\\0\\2\end{pmatrix}-\begin{pmatrix}5\\1\\3\end{pmatrix}=\begin{pmatrix}-3\\-1\\1\end{pmatrix}$ Thus, S' has coordinates $(-3,-1,1)$. --- $\overrightarrow{DS'}=\begin{pmatrix}-3\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}-4\\-4\\2\end{pmatrix}$ Thus, l' has eqn $\mathbf{r}=\begin{pmatrix}1\\3\\-1\end{pmatrix}+\alpha\begin{pmatrix}2\\2\\-1\end{pmatrix}, \alpha\in\mathbb{R}$.	

Method 3 A vector that is perpendicular to both l and L is $$\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$$ Thus, a vector that is parallel to l' is $$\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}-1\\2\\2\end{pmatrix}=\begin{pmatrix}-6\\-6\\3\end{pmatrix}=-3\begin{pmatrix}2\\2\\-1\end{pmatrix}$$ Thus, l' has eqn $\mathbf{r}=\begin{pmatrix}1\\3\\-1\end{pmatrix}+\beta\begin{pmatrix}2\\2\\-1\end{pmatrix}, \beta\in\mathbb{R}.$ S' lies on l': $\overrightarrow{OS'}=\begin{pmatrix}1+2\beta\\3+2\beta\\-1-\beta\end{pmatrix}$ for some $\beta\in\mathbb{R}.$ $\overrightarrow{SS'}=\begin{pmatrix}1+2\beta\\3+2\beta\\-1-\beta\end{pmatrix}-\begin{pmatrix}5\\1\\3\end{pmatrix}=\begin{pmatrix}-4+2\beta\\2+2\beta\\-4-\beta\end{pmatrix}$ Since $\overrightarrow{SS'}\perp L,\ \overrightarrow{SS'}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0.$ Thus, $\begin{pmatrix}-4+2\beta\\2+2\beta\\-4-\beta\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0 \quad \Rightarrow 2+2\beta+4+\beta=0$ $\Rightarrow 3\beta+6=0$ $\Rightarrow \beta=-2$ So $\overrightarrow{OS'}=\begin{pmatrix}1+2(-2)\\3+2(-2)\\-1-(-2)\end{pmatrix}=\begin{pmatrix}-3\\-1\\1\end{pmatrix}$ Thus, S' has coordinates $(-3,-1,1)$.	

Question 12 (Differential Equations)

(i) By similar triangles, radius of the water surface at time t s is $\frac{0.5}{1}h = \frac{h}{2}$ m $W = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ (by similar triangles) $= \frac{1}{12}\pi h^3$	
(ii) Since no water is added to the funnel, $\frac{\mathrm{d}W}{\mathrm{d}t} = -\pi a^2 v$ $= -\pi a^2\sqrt{2gh}$ $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{12}\pi h^3\right) = -\pi a^2\sqrt{2gh}$ $\frac{1}{4}\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t} = -\pi a^2\sqrt{2g}\cdot h^{\frac{1}{2}}$ $\frac{\mathrm{d}h}{\mathrm{d}t} = -\left(4a^2\sqrt{2g}\right)h^{-\frac{3}{2}}$ or $h^{\frac{3}{2}}\frac{\mathrm{d}h}{\mathrm{d}t} = -4a^2\sqrt{2g}$	
(iii) $\int h^{\frac{3}{2}}\frac{\mathrm{d}h}{\mathrm{d}t}\,\mathrm{d}t = -\int 4a^2\sqrt{2g}\,\mathrm{d}t$ $\frac{2}{5}h^{\frac{5}{2}} = -\left(4a^2\sqrt{2g}\right)t + c$ Since $h = 1$ when $t = 0$, $c = \frac{2}{5}$. Therefore, $\frac{2}{5}h^{\frac{5}{2}} = -\left(4a^2\sqrt{2g}\right)t + \frac{2}{5}$ $h^{\frac{5}{2}} = 1 - \left(10a^2\sqrt{2g}\right)t$ $h = \left[1 - \left(10a^2\sqrt{2g}\right)t\right]^{\frac{2}{5}}$, $k = 10$ and $p = \frac{2}{5}$ (shown)	
(iv) For the funnel to become empty, $h = 0$. $1 - \left(10a^2\sqrt{2g}\right)T = 0$ $T = \frac{1}{10a^2\sqrt{2g}}$	

(v) $h=\left[1-\left(10a^2\sqrt{2g}\right)t\right]^{\frac{2}{5}}=\left(1-\frac{t}{T}\right)^{\frac{2}{5}}$