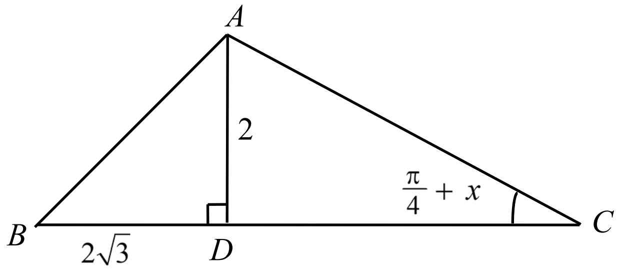
1 A curve has equation $2x+y+2=(x+y)^2+\dfrac{x^2}{1+x^2}$.

Find the equation of the normal to the curve at the point $(0,2)$. [5]

$2x+y+2=(x+y)^2+\dfrac{x^2}{1+x^2}$ $2x+y+2=(x+y)^2+1-\dfrac{1}{1+x^2}\quad ---(1)$ Differentiate (1) w.r.t x: $2+\dfrac{dy}{dx}=2(x+y)\left(1+\dfrac{dy}{dx}\right)-(-1)(1+x^2)^{-2}(2x)\quad ---(2)$ At $(0,2)$, $2+\dfrac{dy}{dx}=2(0+2)\left(1+\dfrac{dy}{dx}\right)+0$ $\dfrac{dy}{dx}=-\dfrac{2}{3}\qquad \Rightarrow$ Gradient of normal is $\dfrac{3}{2}$ Equation of normal at $(0,2)$ is $y=\dfrac{3}{2}x+2$	<u>Be careful when differentiating.</u> Don't forget your chain rule, product rule (or quotient rule), your signs [which can be avoided by adding in appropriate brackets] <u>Be efficient</u> There's no need to make $\dfrac{dy}{dx}$ the subject. You should immediately substitute in $x=0$ and $y=2$

2 The diagram shows triangle ABC, where angle $ACD=\left(\dfrac{\pi}{4}+x\right)$ radians. Point D is on BC such that $AD=2$ and $BD=2\sqrt{3}$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC \approx k(1+\sqrt{3}-2x+2x^2),$$

where k is a constant to be determined. [5]

$BC = BD + DC$ $= 2\sqrt{3}+\dfrac{2}{\tan\left(\dfrac{\pi}{4}+x\right)}$ $= 2\sqrt{3}+\dfrac{2}{\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}}$ $= 2\sqrt{3}+\dfrac{2(1-\tan x)}{1+\tan x}$	<u>Secondary school results:</u> Apply TOA in triangle ADC, we have $DC=\dfrac{2}{\tan\left(\dfrac{\pi}{4}+x\right)}$. No need for sine rule or cosine rule. <u>Maclaurin series expansion</u> #1: If x is small, then $2x$, $3x$ are also small. However $(x+a)$ is not

$\approx 2\sqrt{3}+\dfrac{2(1-x)}{1+x}$ $= 2\sqrt{3}+2(1-x)(1+x)^{-1}$ $= 2\sqrt{3}+2(1-x)[1+(-1)x+\dfrac{(-1)(-2)}{2!})x^2+...]$ $= 2\sqrt{3}+2(1-x)[1-x+x^2+...]$ $= 2\sqrt{3}+2(1-2x+2x^2+...)$ $\approx 2(1+\sqrt{3}-2x+2x^2)$ where $k=2$ **Alternative** Let $f(x)=BC = BD + DC$ $=2\sqrt{3}+\dfrac{2}{\tan\left(\frac{\pi}{4}+x\right)}=2\sqrt{3}+2\cot\left(\frac{\pi}{4}+x\right)$ $f'(x)=-2\operatorname{cosec}^2\left(\frac{\pi}{4}+x\right)$ $f''(x)=-2\left[2\operatorname{cosec}\left(\frac{\pi}{4}+x\right)\right]\left[-\operatorname{cosec}\left(\frac{\pi}{4}+x\right)\cot\left(\frac{\pi}{4}+x\right)\right]$ When $x=0$, $f(0)=2\sqrt{3}+2$ $f'(0)=\dfrac{-2}{\left(\sin\frac{\pi}{4}\right)^2}=-4$ $f''(0)=-2\left(\dfrac{2}{\sin\frac{\pi}{4}}\right)\left[-\dfrac{1}{\sin\frac{\pi}{4}}\cot\left(\frac{\pi}{4}\right)\right]=8$ Hence $BC=2\sqrt{3}+2+(-4)x+\dfrac{8}{2!}x^2+\dots$ $\approx 2\left(\sqrt{3}+1-2x+2x^2\right)$ where $k=2$	considered small, regardless of the size of the constant a. Hence $\tan\left(x+\frac{\pi}{4}\right)\not\approx x+\frac{\pi}{4}$. #2: Since angle x (measured in **radians)** is **small** enough such that x^3 and higher powers of x can be ignored, then $\tan x\approx x$. #3: To find series expansion of $\dfrac{1}{1+x}$, first rewrite into $(1+x)^{-1}$ then use binomial expansion.

3 **(a)** Find $\int (\ln x)^2\ dx$. [3]

$\int (\ln x)^2\ dx = x(\ln x)^2-\int x\cdot 2(\ln x)\cdot\frac{1}{x}\ dx$ $=x(\ln x)^2-2\int \ln x\ dx$ $=x(\ln x)^2-2\left[x\ln x-\int x\cdot\frac{1}{x}\ dx\right]$ $=x(\ln x)^2-2x\ln x+2x+C$	Using by-parts with $u=(\ln x)^2$ and $\frac{dv}{dx}=1$ Hence $\frac{du}{dx}=2(\ln x)\left(\frac{1}{x}\right)$ and $v=x$ 2nd by parts with $u=\ln x$ and $\frac{dv}{dx}=1$ Again don't forget your arbitrary constant.

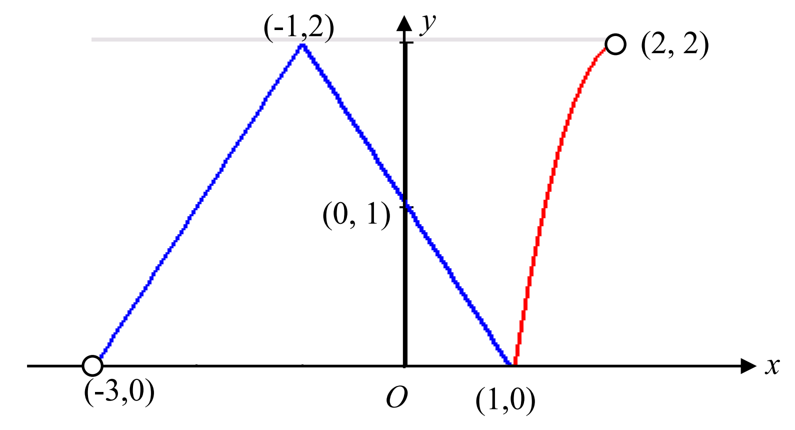
(b) Find $\displaystyle\int \frac{(\sin x+\cos x)^2}{\cos(2x)-2x}\,dx$. [3]

$\displaystyle\int \frac{(\sin x+\cos x)^2}{\cos(2x)-2x}\,dx$ $\displaystyle=\int \frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{\cos(2x)-2x}\,dx$ $\displaystyle=\int \frac{1+\sin 2x}{\cos(2x)-2x}\,dx$ $\displaystyle=-\frac{1}{2}\int \frac{-2\sin 2x-2}{\cos(2x)-2x}\,dx$ $\displaystyle=-\frac{1}{2}\ln\left\|\cos(2x)-2x\right\|+C$	Sometimes when trigonometric functions are involved, you will need to use trigonometric identity to change the form of the integrand in order to apply integration formula.

4 A curve has equation $y=f(x)$, where

$$f(x)=\begin{cases} 2-\|x+1\| & \text{for } -3<x\leqslant 1,\\ 2-2(x-2)^2 & \text{for } 1\leqslant x<2. \end{cases}$$

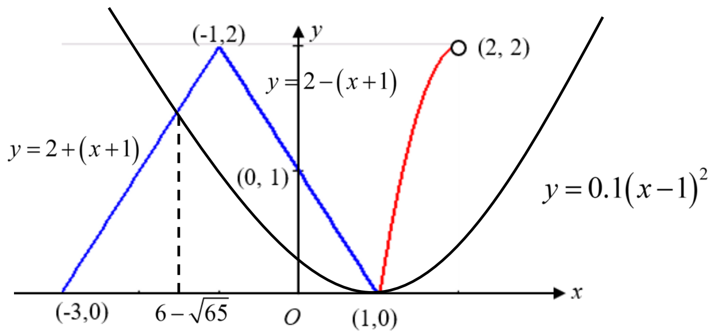
(i) Sketch the curve for $-3<x<2$. [3]



(ii) Hence, solve the inequality $f(x)\leqslant 0.1(x-1)^2$ for $-3<x<2$, leaving your answers in an exact form. [4]

Have to look for points of intersections without using GC

Have to use the graph from (i), to solve the inequality. Hence your solution needs to showcase that method in one way or another.



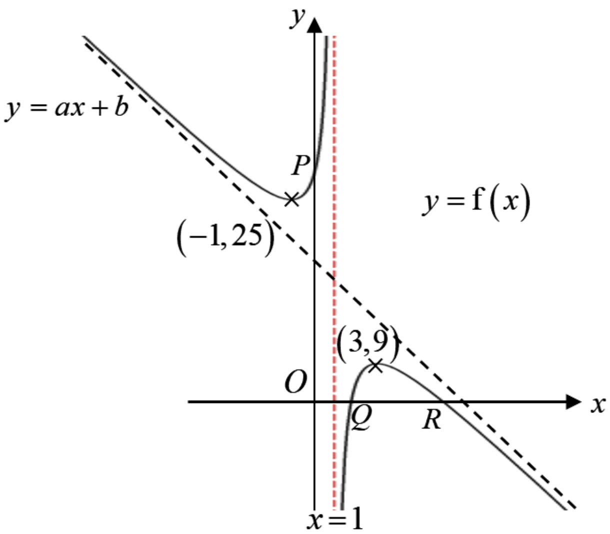
To find the point of intersection: $2+(x+1)=0.1(x-1)^2$ $10x+30=x^2-2x+1$ $x^2-12x-29=0$ $x=\dfrac{12\pm\sqrt{260}}{2}$ $x=6+\sqrt{65}$ or $x=6-\sqrt{65}$ (reject) $\therefore \mathrm{f}(x)\leqslant 0.1(x-1)^2$ $-3<x\le 6-\sqrt{65}$ or $x=1$	Recall definition of modulus function: $\lvert x+1\rvert=\begin{cases} x+1, & \text{when } x\ge -1 \\ -(x+1) & \text{when } x<-1 \end{cases}$ 2 points of intersection: $x=6-\sqrt{65}$ and $x=1$ Note that $-3<x<2$

5 Do not use a calculator in answering this question.
The complex number z satisfies the equation
$$z^2-(4+\mathrm{i})z+2(\mathrm{i}-t)=0,$$
where t is a real number. It is given that one root is of the form $k-k\mathrm{i}$, where k is real and positive. Find t and k, and the other root of the equation. [7]

$(k-k\mathrm{i})^2-(4+\mathrm{i})(k-k\mathrm{i})+2(\mathrm{i}-t)=0$ $\left(k^2-2k(k\mathrm{i})+(k\mathrm{i})^2\right)-\left(4k+k\mathrm{i}-4k\mathrm{i}-k\mathrm{i}^2\right)+2(\mathrm{i}-t)=0$ $\left(k^2-2k^2\mathrm{i}-k^2\right)-\left(4k-3k\mathrm{i}+k\right)+2(\mathrm{i}-t)=0$ $\left(-5k-2t\right)+\mathrm{i}\left(2+3k-2k^2\right)=0$ Compare Real part: $-5k-2t=0$ $---(1)$ Imaginary parts: $2+3k-2k^2=0---(2)$ From (2): $-2k^2+3k+2=0$ $(2k+1)(k-2)=0$ $k=-\dfrac{1}{2}$ or $k=2$ (reject) Substitute $k=2$ into (1), $t=-5$	<u>Secondary school result</u> $(a-b)^2=a^2-2ab+b^2$

$z^2-(4+\mathrm{i})z+2(\mathrm{i}+5)=0$ $z=\dfrac{(4+\mathrm{i})\pm\sqrt{(4+\mathrm{i})^2-4(1)(2\mathrm{i}+10)}}{2}$ $=\dfrac{(4+\mathrm{i})\pm\sqrt{15+8\mathrm{i}-(8\mathrm{i}+40)}}{2}$ $=\dfrac{(4+\mathrm{i})\pm\sqrt{-25}}{2}$ $=\dfrac{4+\mathrm{i}\pm5\mathrm{i}}{2}$ $=2+3\mathrm{i}$ or $2-2\mathrm{i}$ Hence the other root is $2+3\mathrm{i}$ **<u>Alternative (to find other root)</u>** Let $z_1=2-2\mathrm{i}$ and $z_2=$ other root. By sum of roots, $z_1+z_2=-\left(\dfrac{-4-\mathrm{i}}{1}\right)$ $2-2\mathrm{i}+z_2=4+\mathrm{i}$ $z_2=4+\mathrm{i}-2+2\mathrm{i}=2+3\mathrm{i}$	<u>Secondary school result:</u> For any quadratic equation $ax^2+bx+c=0$, Sum of roots, $\alpha+\beta=-\dfrac{b}{a}$ Product of roots, $\alpha\beta=\dfrac{c}{a}$

6



It is given that $\mathrm{f}(x)=ax+b+\dfrac{c}{x-1}$, where a, b and c are constants. The diagram shows the curve with equation $y=\mathrm{f}(x)$. The curve crosses the axes at points P, Q and R, and has stationary points at $(-1,\ 25)$ and $(3,\ 9)$.

Find the values of the constants a, b and c. [4]

$y=\mathrm{f}(x)$ passes through $(3,\ 9)$:

$a(3)+b+\dfrac{c}{3-1}=9 \qquad \Rightarrow 3a+b+\dfrac{1}{2}c=9---(1)$

$y=\mathrm{f}(x)$ passes through $(-1,25)$:

$a(-1)+b+\dfrac{c}{-1-1}=25 \quad \Rightarrow -a+b-\dfrac{1}{2}c=25---(2)$

$\mathrm{f}'(x)=a-\dfrac{c}{(x-1)^2}$

At stationary point $(3,9)$:

$a-\dfrac{c}{(3-1)^2}=0 \quad \Rightarrow a-\dfrac{1}{4}c=0---(3)$

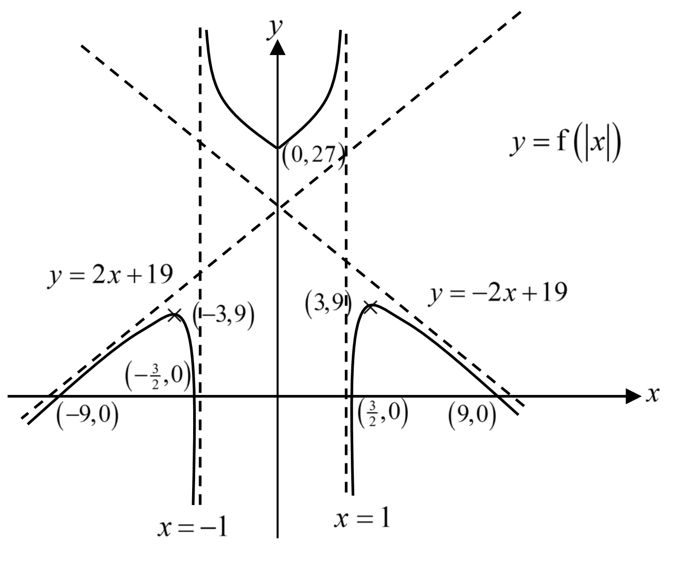
Solving (1), (2) and (3),

$a=-2,\quad b=19,\quad c=-8$

It is now given that points P, Q and R have coordinates $(0,27)$, $\left(\dfrac{3}{2},0\right)$ and $(9,0)$ respectively.

Sketch the curve

(i) $y=\mathrm{f}(|x|)$, [2]



Note that $(0,27)$ is **not** a turning point, hence ensure shape at that point is sketched "sharply"

Observe that the line (oblique asymptote) will be reflected in the y-axis, hence to get the equation of the reflected line, just replace x by $-x$ and we have $y=2x+19$

(ii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

stating the equations of any asymptotes, the coordinates of any points where the curve crosses the axes and of any turning point(s).

Take note of these requirements which applied to both (i) and (ii).

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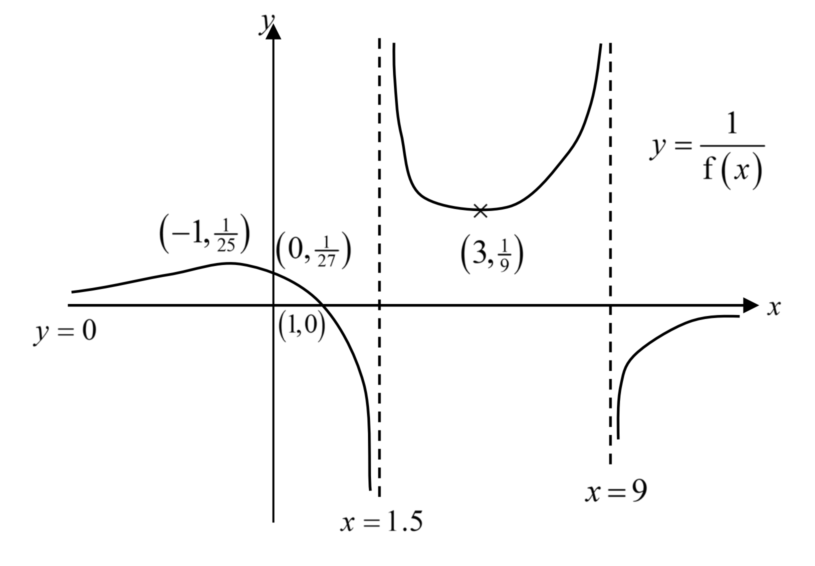

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<td>
$(0,27)\to\left(0,\frac{1}{27}\right)$
$(-1,25)\to\left(-1,\frac{1}{25}\right)$
$(3,9)\to\left(3,\frac{1}{9}\right)$
Take note that $\left(3,\frac{1}{9}\right)$ should be sketched higher than $\left(-1,\frac{1}{25}\right)$
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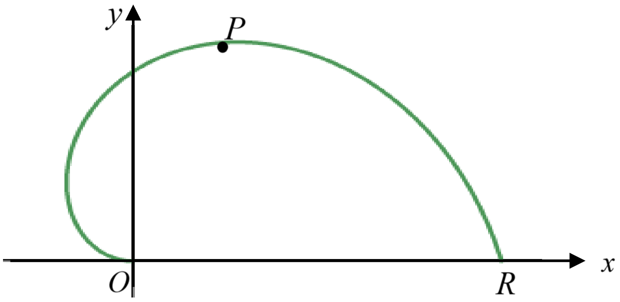
7 The diagram below shows the curve C with parametric equations given by

$$x=-3\theta\cos 3\theta,\quad y=4\theta\sin 3\theta,\qquad \text{for } 0\le\theta\le\frac{\pi}{3}.$$

Point P lies on C with parameter θ and C crosses the x-axis at the origin O and the point R.



(a) Find the area of the region bounded by C and the line $y=\frac{2\pi}{3}-\frac{2x}{3}$, giving your answer correct to 2 decimal places. [4]

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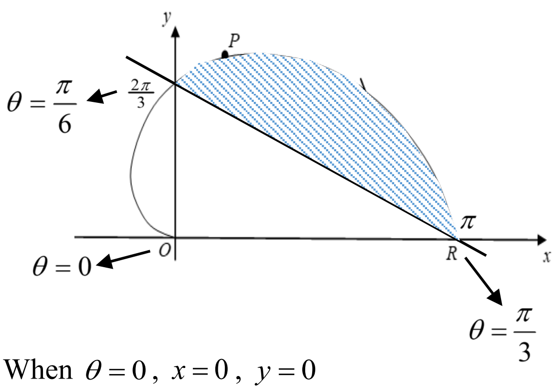

When $\theta=0$, $x=0$, $y=0$

When $\theta=\frac{\pi}{3}$, $x=\pi$, $y=0$

Hence $R(\pi,0)$
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First step is to figure out how to add the line $y=\frac{2\pi}{3}-\frac{2x}{3}$ onto the curve in order to visualise the required region,

Start by finding the coordinates of the x and y-intercepts of both line and C.

Shaded region = region bounded by C and the straight line.
Hence area of required region = area bounded by curve from O to R – (area of triangle)
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To find y-intercept: Let $x=-3\theta\cos 3\theta=0$ $\cos 3\theta=0$ or $\theta=0$ $3\theta=\frac{\pi}{2}$ $\theta=\frac{\pi}{6}$ When $\theta=\frac{\pi}{6}$, $y=4\left(\frac{\pi}{6}\right)\sin 3\left(\frac{\pi}{6}\right)=\frac{2\pi}{3}$ Hence C cuts the y-axis at O and $\left(0,\frac{2\pi}{3}\right)$. Required area $=\int_0^{\pi} y\,\mathrm{d}x-\frac{1}{2}(\pi)\left(\frac{2\pi}{3}\right)$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(4\theta\sin 3\theta)(9\theta\sin 3\theta-3\cos 3\theta)\,\mathrm{d}\theta-\frac{\pi^2}{3}$ $=2.74$ units^2	$y=4\theta\sin 3\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta}=9\theta\sin 3\theta-3\cos 3\theta$, When $x=0$, ($y=\frac{2\pi}{3}$), hence $\theta=\frac{\pi}{6}$ When $x=\pi$, $\theta=\frac{\pi}{3}$

(b) Use differentiation to find the maximum value of the area of triangle OPR as θ varies, proving that it is a maximum. [6]

Rigor is expected i.e. $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=-50.968$ must be seen. If the first derivative test is used, then $\frac{\mathrm{d}A}{\mathrm{d}\theta}$ has to be evaluated at the 3 values of θ

Note $P(-3\theta\cos 3\theta,\ 4\theta\sin 3\theta)$ Let A be the area of the triangle OPR $A=\frac{1}{2}\times\pi\times 4\theta\sin 3\theta=2\pi\theta\sin 3\theta$ $\frac{\mathrm{d}A}{\mathrm{d}\theta}=2\pi[3\theta\cos 3\theta+\sin 3\theta]$ For max A, $\frac{\mathrm{d}A}{\mathrm{d}\theta}=0$ $2\pi[3\theta\cos 3\theta+\sin 3\theta]=0$ By GC, $\theta=0, 0.67625$ or $\theta=0$ (reject) $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=2\pi[-9\theta\sin 3\theta+3\cos 3\theta+3\cos 3\theta]$ $=6\pi[2\cos 3\theta-3\theta\sin 3\theta]$ Substituting $\theta=0.67625$ $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=6\pi[2\cos(3\times 0.67625)-3\theta\sin(3\times 0.67625)]$ $=-50.968<0$ Maximum $A=2\pi\times 0.67625\times\sin(3\times 0.67625)$ $=3.81$ (3s.f.)	Note that O and R are fixed points hence $OR=\pi$

8 (a) The sum of the first n terms of a sequence is given by $S_n = \dfrac{n^2+5n}{8}$. Show that the sequence follows an arithmetic progression with common difference d, where d is to be determined.
In a geometric progression, the first term is 100 and its common ratio is $3d$. Find the smallest value of k such that the sum of the first k terms of the arithmetic progression is greater than the sum of the first 30 terms of the geometric progression. [6]

$u_n = S_n - S_{n-1}$ $= \dfrac{n^2+5n}{8} - \dfrac{(n-1)^2+5(n-1)}{8}$ $= \dfrac{1}{8}\left[n^2+5n-\left(n^2-2n+1+5n-5\right)\right]$ $= \dfrac{1}{4}(n+2)$	$u_n = S_n - S_{n-1}$ (not $S_{n+1} - S_n$) is true for all series Be extra careful with algebraic manipulations/expansions, e.g. $-5(n-1) = -5n+5$, not $-5n-5$, not $-5n+1$
Consider $u_n - u_{n-1}$ $= \dfrac{1}{4}(n+2) - \dfrac{1}{4}(n-1+2)$ $= \dfrac{1}{4}$ = constant Hence, the sequence is an arithmetic progression with common difference, $d = \dfrac{1}{4}$	Conclude properly as it is a 'show' question.
$\dfrac{k^2+5k}{8} > \dfrac{100\left(1-\left(\frac{3}{4}\right)^{30}\right)}{1-\frac{3}{4}}$ $k^2+5k-3199.429 > 0$ $(k-54.119)(k+59.119) > 0$ $k < -59.119$ or $k > 54.119$ Since $k \geq 0$, smallest $k = 55$	Expression for LHS, i.e. $\dfrac{k^2+5k}{8}$ is already stated in the question. There is no need to rewrite it using $S_n = \dfrac{n}{2}\left[2a+(n-1)d\right]$ Show clearly how the **inequality** is solved Answer is <u>**smallest**</u> k = 55, not k = 55

(b) The first and second terms of a geometric sequence are $u_1 = a$ and $u_2 = a^2 - a$. If all the terms of the sequence are positive, find the set of values of a for which $\sum_{r=1}^{\infty} u_r$ converges. [2]

Common ratio $r = \frac{u_2}{u_1} = a - 1$ For the series to converge, $\lvert r \rvert < 1$ $\Rightarrow -1 < a - 1 < 1$ $\Rightarrow 0 < a < 2$ Furthermore, for all terms to be positive, $u_1 = a > 0$ and $r = a - 1 > 0$ Alternatively, For the series to converge, <u>and</u> all terms to be positive $0 < r < 1$ $\Rightarrow 0 < a - 1 < 1$ $\Rightarrow 1 < a < 2$ Hence, the set of values of a $\{a \in \mathbb{R} : 1 < a < 2\}$	Read the question carefully, <u>all terms</u> are positive, hence first term and r must be positive. Give your final answer in set notation

For this sequence, it is known that the sum of all the terms after the nth term is equal to the nth term. Find the value of a and hence the value of $\sum_{r=1}^{\infty} u_r$. [3]

Given : $u_n = u_{n+1} + u_{n+2} + \ldots$ $\Rightarrow ar^{n-1} = ar^n + ar^{n+1} + \ldots.$ $\Rightarrow \frac{ar^n}{r} = \frac{ar^n}{1-r}$ $\Rightarrow r = 1 - r$ $\Rightarrow r = \frac{1}{2}$ i.e. $a = \frac{3}{2}$ Thus $S_\infty = \frac{a}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$ Alternatively, Using $n = 1$ $S_\infty - a = a$ $\frac{a}{1-(a-1)} - a = a$ $a = \frac{3}{2}$ Thus $S_\infty = \frac{a}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$	Read the question carefully, "… sum of all terms <u>**after**</u> the nth term…"

9 The curve C has equation

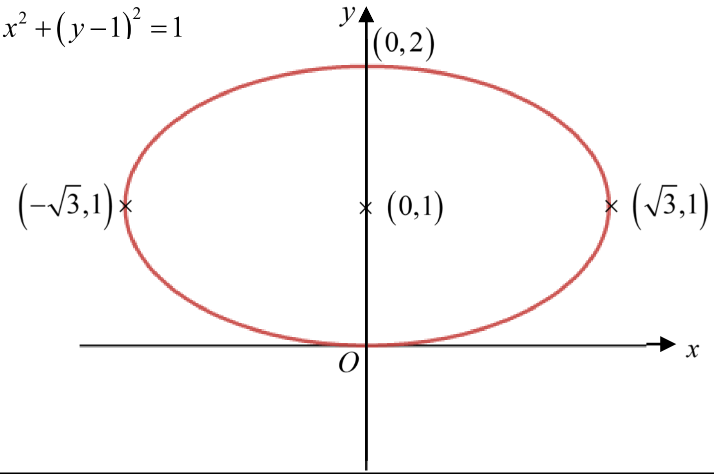
$$\frac{1}{3}x^2+y^2-2y=0.$$

(i) Sketch C. [2]

$$\frac{1}{3}x^2+y^2-2y=0$$

$$\frac{1}{3}x^2+(y-1)^2-1=0$$

$$\frac{1}{3}x^2+(y-1)^2=1$$



Need to label the coordinates of all vertices, including centre.

Shape of ellipse is  not  as $\sqrt{3}>1$.

(ii) Use the substitution $x=p\sin\theta$ to show that

$$\int_0^p \sqrt{p^2-x^2}\,\mathrm{d}x = \frac{p^2\pi}{4},$$

where p is a positive constant. [4]

$$\int_0^p \sqrt{p^2-x^2}\,\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}} \sqrt{p^2-p^2\sin^2\theta}\,(p\cos\theta)\,\mathrm{d}\theta$$

$$=p^2\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\,(\cos\theta)\,\mathrm{d}\theta$$

$$=p^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta \qquad \left[\cos\theta\ge 0 \quad \text{for } 0\le\theta\le\frac{\pi}{2}\right]$$

$$=p^2\int_0^{\frac{\pi}{2}}\left(\frac{\cos 2\theta+1}{2}\right)\mathrm{d}\theta$$

$$=p^2\left[\frac{\sin 2\theta}{4}+\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$=p^2\left(\frac{\sin 2\left(\frac{\pi}{2}\right)}{4}+\frac{\frac{\pi}{2}}{2}\right)=\frac{p^2\pi}{4}$$

$x=p\sin\theta$

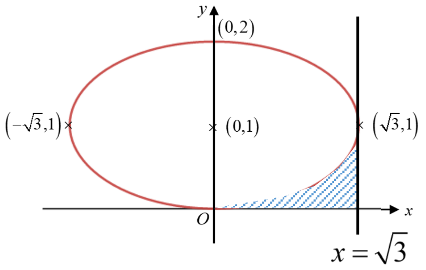
$\mathrm{d}x=p\cos\theta\,\mathrm{d}\theta$

When $x=0$, $\theta=0$

When $x=p$, $\theta=\frac{\pi}{2}$

Remember to **change the limits** for definite integrals using substitution

(iii) The region R is bounded by C, the line $x=\sqrt{3}$ and the x-axis. Find the exact area of R. [3]

 $\frac{1}{3}x^2+(y-1)^2=1$ $y=1-\sqrt{1-\frac{1}{3}x^2}\quad (\because y<1)$ Area $=\int_0^{\sqrt{3}}\left(1-\sqrt{1-\frac{1}{3}x^2}\right)\mathrm{d}x$ $=\int_0^{\sqrt{3}}1\,\mathrm{d}x-\frac{1}{\sqrt{3}}\int_0^{\sqrt{3}}\sqrt{3-x^2}\,\mathrm{d}x$ $=\sqrt{3}-\frac{1}{\sqrt{3}}\left(\frac{3\pi}{4}\right)$ $=\sqrt{3}-\frac{\sqrt{3}\pi}{4}$	It is very useful to do a simple sketch and shade the region Do not always assume positive square root for all cases. For this section of the curve, $0\le y\le 1$, so $y=1-\sqrt{1-\frac{1}{3}x^2}$ Always check if you can use the previous part to solve. In this case, $\int_0^{\sqrt{3}}\sqrt{1-\frac{1}{3}x^2}\,\mathrm{d}x$ looks close to the previous result $\int_0^p\sqrt{p^2-x^2}\,\mathrm{d}x=\frac{p^2\pi}{4}$ <u>Exact</u> answer is required, so one cannot use GC.

(iv) R is rotated completely about the y-axis. Find the exact volume of the solid obtained. [3]

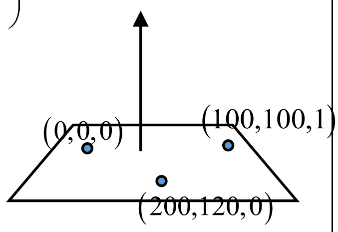
Volume $=\pi\left(\sqrt{3}\right)^2(1)-\pi\int_0^1 x^2\,\mathrm{d}y$ $=3\pi-\pi\int_0^1\left(6y-3y^2\right)\mathrm{d}y$ $=3\pi-\pi\left[3y^2-y^3\right]_0^1$ $=\pi$	Read the question carefully – R is rotated about <u>**y -axis,**</u> not x-axis. Most students who produced a sketch got this part correct by realising the solid is a **hollow** figure.

(ii) Describe a pair of transformations which transforms the graph of C onto the graph of $x^2+y^2=1$. [2]

Translate 1 units in the negative y-direction. Stretch the resultant curve by a factor of $\frac{1}{\sqrt{3}}$ parallel to the x-axis, y-axis invariant.	

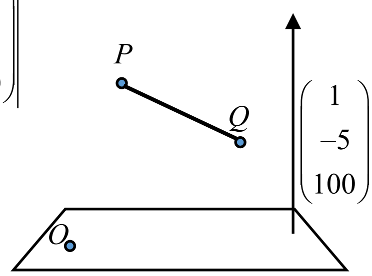
10 Workers are installing zip lines at an adventure park. The points (x, y, z) are defined relative to the entrance at $(0,0,0)$ on ground level, where units are in metres. The ticketing booth at $(100,100,1)$ and lockers at $(200,120,0)$ are also on ground level. Zip lines are laid in straight lines and the widths of zip lines can be neglected. The ground level of the park is modelled as a plane.

(i) Find a cartesian equation of the plane that models the ground level of the park. [2]

A normal to the ground, $\underset{\sim}{n} = \begin{pmatrix}100\\100\\1\end{pmatrix} \times \begin{pmatrix}200\\120\\0\end{pmatrix}$ $= \begin{pmatrix}-120\\200\\-8000\end{pmatrix}$ $= -40\begin{pmatrix}3\\-5\\200\end{pmatrix}$  Equation of ground: $3x - 5y + 200z = 0$	Try to avoid unnecessary careless mistakes in cross product by writing clearer and bigger Question asked for "**<u>cartesian</u>** equation"

A zip line connects the points $P(300,120,30)$ and $Q(300,320,25)$, and is modelled as a segment of the line l. The façade of a building nearby can be modelled as part of the plane with equation $\mathbf{r}.\begin{pmatrix}1\\-5\\100\end{pmatrix} = 0$. As a safety requirement, every point on the zip line must be at a distance of at least 10 metres away from the façade of the building.

(ii) Write down a vector equation of l. Hence, or otherwise, determine if the zip line passes the safety requirement. [4]

$l: \quad \underset{\sim}{r} = \begin{pmatrix}300\\120\\30\end{pmatrix} + \lambda\begin{pmatrix}0\\-200\\5\end{pmatrix}, \quad \lambda \in \mathbb{R}$ Note that at P, $\lambda = 0$; and at Q, $\lambda = -1$ and that the origin lies on the plane of the façade. Distance from point on zip line to façade $= \left\lvert \left[\begin{pmatrix}300\\120\\30\end{pmatrix} + \lambda\begin{pmatrix}0\\-200\\5\end{pmatrix}\right] . \frac{1}{\sqrt{10026}}\begin{pmatrix}1\\-5\\100\end{pmatrix} \right\rvert$ $= \frac{\lvert 2700 + 1500\lambda \rvert}{\sqrt{10026}}$ $\geq \frac{1200}{\sqrt{10026}} \qquad \because -1 \leq \lambda \leq 0$ $\approx 11.984 > 10$  Since all points on the zip line are more than 10 m away from the façade of the building, it passes the safety requirement.	Note the zip line is modelled as a **segment** PQ, not the whole line l. One needs to consider the distance from **a general point on the line segment *PQ*** to the plane. It is not sufficient to find the distance from P (or Q) to the plane.

Alternatively,
Distance from P to façade

$$= \left| \begin{pmatrix} 300 \\ 120 \\ 30 \end{pmatrix} . \frac{1}{\sqrt{10026}} \begin{pmatrix} 1 \\ -5 \\ 100 \end{pmatrix} \right|$$

$$= \frac{2700}{\sqrt{10026}}$$

$$= 27.0 > 10$$

Distance from Q to façade

$$= \left| \begin{pmatrix} 300 \\ 320 \\ 25 \end{pmatrix} . \frac{1}{\sqrt{10026}} \begin{pmatrix} 1 \\ -5 \\ 100 \end{pmatrix} \right|$$

$$= \frac{1200}{\sqrt{10026}}$$

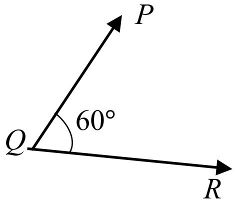
$$= 12.0 > 10$$

Since P and Q are both on the same side of the building façade, all points on the zip line are more than 10 m away from the façade of the building, it passes the safety requirement.

The workers need to install another zip line from Q to $R(127, 220, a)$, where $0 < a < 30$, and the angle PQR is given to be $60°$.

(iii) Find the value of a, leaving your answer to 3 decimal places. [3]

$$\overrightarrow{QP} = \begin{pmatrix} 0 \\ -200 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 0 \\ -40 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{QR} = \begin{pmatrix} -173 \\ -100 \\ a-25 \end{pmatrix}$$



$$\cos 60^{\circ} = \frac{\begin{pmatrix} 0 \\ -40 \\ 1 \end{pmatrix} . \begin{pmatrix} -173 \\ -100 \\ a-25 \end{pmatrix}}{\sqrt{1601}\sqrt{39929 + (a-25)^2}}$$

$$\left(\frac{1}{4}\right)(1601)\left(39929 + (a-25)^2\right) = \left(4000 + (a-25)\right)^2$$

$$399.25(a-25)^2 - 8000(a-25) - 18417.75 = 0$$

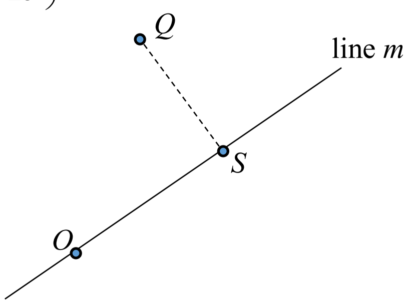
By GC, $a - 25 = -2.08522$ or $a - 25 = 22.1228$ (rej. $\because a < 30$)
$a = 22.915$ (to 3 d.p.)

Recall definition of dot product
$\overrightarrow{QP} \bullet \overrightarrow{QR} = \left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right| \cos\theta$
with <u>both vectors either outward facing or inward facing</u>.

Question reads "to 3 decimal places", so it is a clue to **use GC** to solve an equation with only one unknown.

The façade of the building meets the ground level of the park at line m. A worker sets up a transmitter at point S on line m such that S is nearest to Q.

(iv) Find a vector equation of m and the distance from Q to S. [4]

Façade: $x-5y+100z=0$ Ground: $3x-5y+200z=0$ Solving simultaneously by GC, m: $\underset{\sim}{r}=\mu\begin{pmatrix}-50\\10\\1\end{pmatrix}, \quad \mu\in\mathbb{R}$ Since O lies on m, and $\overrightarrow{OQ}=\begin{pmatrix}300\\320\\25\end{pmatrix}$.  $QS=\dfrac{\left\lvert\begin{pmatrix}300\\320\\25\end{pmatrix}\times\begin{pmatrix}-50\\10\\1\end{pmatrix}\right\rvert}{\sqrt{2601}}$ $=\dfrac{\left\lvert\begin{pmatrix}70\\-1550\\19000\end{pmatrix}\right\rvert}{\sqrt{2601}}$ $=\dfrac{\sqrt{363407400}}{\sqrt{2601}}$ ≈ 374 The distance from Q to S is 374 m (to 3 s.f.).	Use GC to solve 2 cartesian equations simultaneously Not necessary to add $\begin{pmatrix}0\\0\\0\end{pmatrix}$ in the equation of m. $\underset{\sim}{r}=\cancel{\begin{pmatrix}0\\0\\0\end{pmatrix}}+\mu\begin{pmatrix}-50\\10\\1\end{pmatrix}$

11 A tank contains 500 litres of water in which 100 g of a poisonous chemical called Prokrastenate is dissolved. A solution containing 0.1 g of Prokrastenate per litre is pumped into the tank at a rate of 5 litres per minute, and the well-mixed solution is pumped out at the same rate. By letting x grams be the amount of Prokrastenate in the tank after t minutes, show that

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{k-x}{100},$$

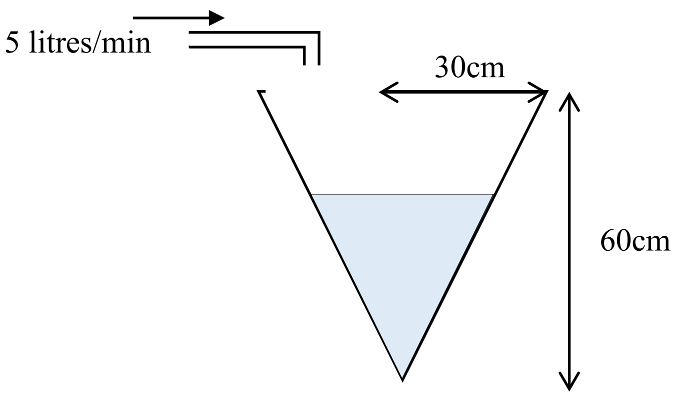
where k is a constant to be determined. [2]

$\dfrac{\mathrm{d}x}{\mathrm{d}t}=0.1\times5-\dfrac{x}{500}\times5$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=0.5-\dfrac{x}{100}$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{50-x}{100}$	

Find x in terms of t and find the ~~time taken for a quarter of Prokrastenate to be removed from the tank.~~ value of t when $x = 75$. [5]

$\frac{1}{50-x}\frac{dx}{dt}=\frac{1}{100}$ $\int \frac{1}{50-x}dx=\int \frac{1}{100}dt$ $-\ln\lvert 50-x\rvert=\frac{t}{100}+C$ $50-x=Ae^{-\frac{t}{100}}$ $x=50-Ae^{-\frac{t}{100}}$ Substituting $t=0, x=100 \Rightarrow A=-50$ Hence, $x=50+50e^{-\frac{t}{100}}$ Substitute $x=75$, $75=50+50e^{-\frac{t}{100}}$ $e^{-\frac{t}{100}}=0.5$ $t=100\ln 2=69.3$	One should always attempt to solve the DE even if k was not found in the earlier part. Evaluate the final answer as 69.3 (3sf) and not leave it as 100ln2.

The well-mixed solution is that is pumped out flows into an empty container, in the form of an open inverted cone with a height of 60 cm and base radius 30 cm, at the same rate (see diagram).



Given that 1 litre = 1000 cm^3,

(i) Show that the volume of the well-mixed solution in the container, V cm^3 can be expressed as $V=\frac{\pi h^3}{12}$, where h cm is the depth of the solution at that instant. [2]

Let the radius and height of water after t min be r cm and h cm respectively. By similar triangles, $\frac{r}{h}=\frac{30}{60}$ $r=\frac{h}{2}$	Need to **explain** $r=\frac{h}{2}$ using similar triangles as it is a 'show' question

$V=\frac{1}{3}\pi r^2 h$ $=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi h^3}{12}$	

(ii) Hence, or otherwise, find the rate of change of the depth of solution after 5 minutes. [4]

[The volume of a cone of base radius r and height h is given by $V=\frac{1}{3}\pi r^2 h$.]

After 5 min, $5\times5\times1000=\frac{\pi h^3}{12}$ $h=\sqrt[3]{\frac{30000}{\pi}}=45.708$ $V=\frac{\pi h^3}{12}$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{\pi h^2}{4}$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $5000=\frac{\pi(45.708)^2}{4}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=3.0472$ The height of the water level is increasing at the rate 3.05 cm/ min after 5 min.	Convert **5 litres to 5000cm^3** in order to be consistent in the use of units, since cm is used in the question Final answer is in **cm/min**, not cm/s