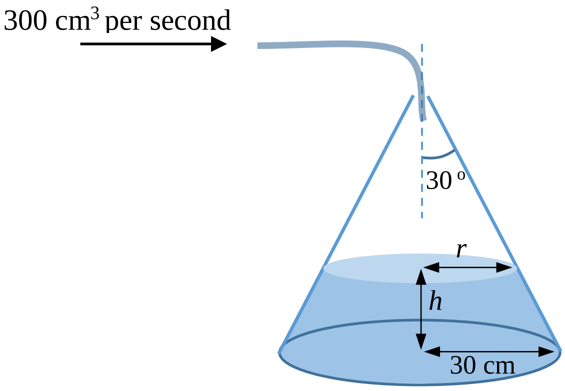
**1** Water is poured at a constant rate of 300 cm$^3$ per second into a conical container, as shown in the diagram. The conical container has a base radius of 30 cm and semi-vertical angle of 30°. At time $t$ seconds, the water in the conical container has a volume of $V$ cm$^3$, depth $h$ cm and the radius of the water surface $r$ cm.




**(i)** Show that $V = 9000\sqrt{3}\pi - \frac{\pi}{9}\left(30\sqrt{3} - h\right)^3$. [3]

[The volume of cone with radius $r$ and height $h$ is $\frac{1}{3}\pi r^2 h$.]

**(ii)** Find the rate of change of depth of the water in the conical container when $h = 5\sqrt{3}$. [2]

**2** Solve the inequality $\frac{x^2 + 2x + 3}{ax^2 - (a+1)x + 1} < 0$, for $a \in \mathbb{R}$, $a \neq 0$, $a < 1$. You need to distinguish the cases where $a$ is positive and $a$ is negative. [5]

**3** It is given that $\mathrm{f}(r) = \ln\left(2 + \alpha^{r+1}\right)$, where $\alpha$ is a constant and $-1 < \alpha < 1$.

By considering $\mathrm{f}(r+1) - \mathrm{f}(r)$, find $\sum_{r=1}^{n} \ln\left(\frac{2+\alpha^{r+2}}{2+\alpha^{r+1}}\right)$ in terms of $n$ and $\alpha$. [3]

Hence, give a reason why the series

$$\ln\left(\frac{2+\alpha^4}{2+\alpha^3}\right) + \ln\left(\frac{2+\alpha^5}{2+\alpha^4}\right) + \ln\left(\frac{2+\alpha^6}{2+\alpha^5}\right) + \ldots$$

converges. Given that $\alpha = 0.7$, find the value of the sum to infinity. [4]

**[Turn Over**

**4** **(i)** It is given that $y=\sqrt{1+\ln(1+2x)}$. Show that $y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+2x}$. [2]

**(ii)** By repeated differentiation of this result, find the Maclaurin expansion of $y$ in ascending powers of $x$, up to and including the term in $x^2$. [3]

**(iii)** Verify that the same result is obtained using the standard series expansions given in the List of Formulae (MF26). [3]

**5** **(i)** One of the roots of the equation $x^3-3x+c=0$, where $c$ is real, is $2-3\mathrm{i}$.
Without the use of a calculator, find the value of $c$ and the other roots of the equation. [5]

**(ii)** If $c$ is a non-zero purely imaginary number, explain if it is possible for $x^3-3x+c=0$ to have real roots. [1]

**(iii)** It is given instead that $c$ is a real number. By considering the graph of $y=x^3-3x$, find the range of values of $c$ for which the equation $x^3-3x+c=0$ has only real roots. [2]

**6** The sequence $a_1, a_2, a_3...$ is a geometric progression $A$ with common ratio $\frac{3}{4}$, and the sequence $b_1, b_2, b_3...$ is an arithmetic progression $B$. The sum to infinity of $A$ is equal to the sum of the first ten terms of $B$. Given that $a_1=b_9+13$ and the sum of $a_2$ and $b_2$ is 35.825, find $a_1+a_2+a_3+...+a_{25}$, giving your answer correct to 2 decimal places. [8]

**7** With reference to the origin $O$, the points $A$, $B$, $P$, $Q$ and $R$ have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}-2\mathbf{b}$, $-2\mathbf{a}-3\mathbf{b}$ and $2\mathbf{a}+\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors and $\mathbf{a}\bullet\mathbf{b}<0$.

Given that $\mathbf{a}$ is a unit vector and the area of triangle $PQR$ is equal to the magnitude of $\mathbf{b}$, show that $\sin\theta=\frac{1}{4}$, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Hence, find the value of $\theta$. [5]

$M$ lies on $PR$ such that $PM=\frac{1}{2}MR$. Given that $PR$ is perpendicular to $OM$, find the magnitude of $\mathbf{b}$, giving your answer correct to 3 decimal places. [5]

**8** **(a)** The curve $C$ with equation $y=\frac{x^2+\lambda}{x+2}$, $\lambda\neq 4$, where $\lambda$ is a real constant, has a positive gradient at any point on the curve.

**(i)** Find the range of values of $\lambda$. [2]

**(ii)** Sketch $C$, stating the equations of any asymptotes and the coordinates of the points where $C$ crosses the axes. [3]

**(b)** The transformations A, B and C are given as follows:

A: A translation of 3 units in the negative $x$-direction.

B: A reflection about the $x$-axis.

C: A stretch parallel to the $y$-axis with a stretch factor of 4, with $x$-axis invariant.

A curve undergoes in succession, the transformations A, B and C and the equation of the resulting curve is $y=-\frac{4(x+3)}{x+2}$.

Determine the equation of the curve before the transformations were effected. [3]

**(c)** It is given that

$$\mathrm{f}(x)=\begin{cases} x & \text{for } 0\le x\le 1, \\ \sqrt{1-\frac{(x-1)^2}{4}} & \text{for } 1<x\le 3, \end{cases}$$

and that $\mathrm{f}\left(x\right)=\mathrm{f}\left(x+3\right)$ for all real values of *x*.

On separate diagrams, sketch for $-2\leqslant x\leqslant 4$, the graphs of

**(i)** $y=\mathrm{f}(x)$, [3]

**(ii)** $y=\mathrm{f}(|x|)$. [2]

**9** The position of a particle *P*, moving along a curve *C*, at any time *t* is given by the parametric equation

$$x=\frac{t}{2}-\sin t, \quad y=3-\cos t, \text{ for } 0\leqslant t\leqslant \frac{3\pi}{2},$$

where *x* and *y* are measured in metres and *t* in seconds.

**(i)** Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\sin t}{1-2\cos t}$. [2]

**(ii)** Find the exact equation of the tangent to *C* at which the tangent is parallel to the *y*-axis. [3]

**(iii)** Sketch the graph of *C*. Give in exact form the coordinates of the points where *C* meets the *y*-axis, and also give in exact form the coordinates of the end points and maximum point on the curve. [4]

**(iv)** The speed of the particle at time *t* is given by the formula

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}.$$

Given that the particle is moving at maximum speed when $t=\pi$, find its speed at this instance. [1]

**(v)** The distance between two points along a curve is the arc length. The arc length between two points on *C*, where $t=\alpha$ and $t=\beta$, is given by the formula

$$\int_{\alpha}^{\beta}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Find the distance, in metres, travelled by the particle from the start to the instant when it attains maximum speed, giving your answer correct to 3 decimal places. [2]

**0** The velocity, $v$ of an object is given by $v=\frac{\mathrm{d}x}{\mathrm{d}t}$, where $x$ is its displacement from a point $O$ at time $t$. Given that the acceleration, $a$ of the object is given by $a=\frac{\mathrm{d}v}{\mathrm{d}t}$, prove that $a=v\frac{\mathrm{d}v}{\mathrm{d}x}$. [1]

Newton's second law states that the *nett force* (in N) acting on an object is equal to the product of its mass (in kg) and its acceleration (in $\mathrm{ms}^{-2}$).

In military exercises, parachutists jump from stationary helicopters and their motion is tracked by sensors tagged to their bodies. One parachutist of mass 80 kg falls vertically from a Chinook helicopter. When the parachutist is $x$ m below the helicopter (when the parachute is not opened), his velocity is $v\ \mathrm{ms}^{-1}$ and the nett force acting on him is $800-0.4v^2$.

Show that his motion can be modelled by the differential equation $v\frac{\mathrm{d}v}{\mathrm{d}x}=10-0.005v^2$. [2]

Solve the differential equation and sketch $v$ against $x$ where $v\geq 0$. [8]

For an object falling through the atmosphere, the object is said to have reached *terminal velocity* when the object's acceleration is zero. State the *terminal velocity* of the parachutist. [1]

**11** Let $\theta=\mathrm{f}(t)$ be the outdoor temperature, in degree Celsius, of a typical day in May in a small town, $t$ hours after 12 midnight. It can be shown that

$$\mathrm{f}(t)=a\cos\left(\frac{\pi}{24}t-\frac{\pi}{2}\right)+b,\ 0\leqslant t<24 \text{ and } a \text{ and } b \text{ are positive constants.}$$

It is given that on a typical day in May, the outdoor temperature is 25 $°C$ at 12 midnight and the maximum temperature of the day is 38 $°C$ at 12 noon.

**(i)** Find the value $b$ and show that $a=13$. [2]

**(ii)** It is given $\alpha$ and $\beta$, where $\alpha\neq\beta$ are such that $\mathrm{f}(\alpha)=\mathrm{f}(\beta)$. Show that $\alpha+\beta=k$, where $k$ is a constant to be determined. [2]

The rate of absorption of nutrients, $\mathrm{g}(\theta)$ $\mathrm{mgs}^{-1}$, of a plant is influenced by the outdoor temperature $\theta$, and can be modelled by the function g where

$$\mathrm{g}:\theta\mapsto 50\ln\theta-150,\ \theta>23.$$

**(iii)** Write, in context of the question, what the composite function gf represents and show that this function exists. [3]

**(iv)** Determine if gf has an inverse. [2]

**(v)** Find the range of the rate of absorption between (12 – $s$) am and $s$ pm on a typical day in May. [3]